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THE ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY.

BY

EUGENE L. RICHARDS, B. A.,
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PREFACE.

In the following pages the author has aimed to make the subject of Trigonometry plain to beginners, and has, therefore, devoted a great deal of space to the elementary definitions and to the application of them.

Spherical Trigonometry is a subject so difficult to grasp, that any treatise on it designed for beginners should, in the opinion of the writer, make frequent use of diagrams, to convey to the student a clear idea of the relations of the magnitudes under treatment. Consequently, this kind of illustration has been applied wherever it could be of any possible assistance.

The references are to "Todhunter's Euclid" and to "Chauvenet's Geometry."

The answers to examples have been obtained by means of six-place tables.

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THE

ELEMENTS OF PLANE TRIGONOMETRY.

CHAPTER I.

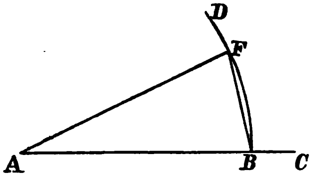
SOME INSTRUMENTS USED IN DRAWING.

FOR the right representation of the magnitudes of which Trigonometry treats, the following instruments will be found useful: 1. A *pair* of *dividers* or *compasses*, having one leg to which can be adjusted a pencil, or, if necessary, an ink-point; 2. A *plane-scale.*

One of the simplest scales, and one which is sufficient for all ordinary drawing purposes, is six inches in length, and is made of ivory or of box-wood. It contains a line or lines of chords, and also lines of equal parts, of which the units are of different lengths.

(a) A *line of chords* is used for setting off angles. It consists of a line, marked from 10° to 90°, and contains the chords of all arcs from 1° up to 90°. As the chord of 60° is equal to radius (Euc. 16, IV. Cor. Ch. 5, V.), when we wish to lay down an angle of a given number of degrees we describe a circle, from a given point, with a radius equal to the length of the line measured from the beginning of the line of chords to the division marked 60; then taking the chord of the number of degrees whose angle is required as a radius, we place it in the circle. If the extremities of the chord be joined to the centre of the circle, the chord will subtend the angle required.

Thus, suppose at a given point in a given line we wish to make an angle of 25°.



Let A be the given point in the given line $A\,C$, at which it is required to make an angle of 25°.

From the line of chords we take a chord of 60° as radius, and with it, from A as a centre, describe an arc $B\,D$. Then from B as a centre, with the line $B\,F$, as a radius, equal to a chord of 25°, taken from the *same* line of chords, we describe an arc intersecting the former arc at F. Join $B\,F$ and $A\,F$. Then $F\,A\,B$ is an angle of 25°, since it is at the centre of the circle, and subtends an arc of 25°.

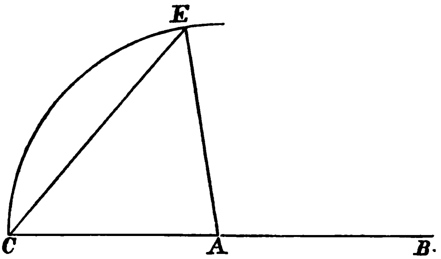
If it is required, at a given point, to make an obtuse angle of a given number of degrees, we produce the line through the given point, and make an angle on the line produced equal to the supplement of the given angle.

Thus let it be required to make an angle of 100° at the point A in the given line $A\,B$.

Produce the line $A\,B$ through A. Make $A\,C$ equal to a chord of 60° taken from the line of chords. From

A as centre, with $A\ C$ as radius, describe an arc $C\ E$. From C as centre, with a radius $C\ E$, equal to a chord of 80°, taken from the *same* line of chords, describe an arc cutting the former arc at E. Then $C\ A\ E$ is an angle of 80°. But as the angles $C\ A\ E$ and $E\ A\ B$ are together equal to two right angles, or 180°, $E\ A\ B$ is equal to 100°.

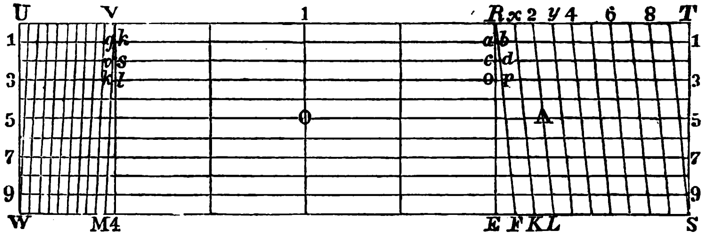
Also an obtuse angle of a certain number of degrees can be made by making two adjacent acute angles, the sum of which is equal to the given obtuse angle.



(Angles of a certain number of degrees and *minutes*, or degrees, *minutes*, and *seconds*, can be made only approximately correct by the line of chords.)

(b) A *line of equal parts* is used for drawing lines of a given length, one of the equal parts being taken as the unit, or as ten units, or as a hundred units, if the line is measured on the decimal system. Lines of equal parts are also given, by means of which feet and inches can be represented. The unit is then divided into twelve equal parts.

To obtain hundredths of the unit on the plane-scale a device is adopted which will be understood by the following figure:



In the above figure, which represents a part of a plane-scale in which the unit of measurement is an inch, let RS be a square of which each side is an inch. Let RT be divided into ten equal parts, Rx, $x2$, etc.; also let ES and ST be divided into ten equal parts. Through the points of division of ST let lines be drawn parallel to RT or ES. Let a straight line be drawn from R to the first point (F) of division in the line ES. Let xk be drawn from the first point of RT to the second point of ES, another line be drawn from the second point of RT to the third point of ES, and so on.

Since Rx, $x2$, $2y$, etc., are equal and parallel to FK, KL, etc., RF, xK, $2L$, etc., are parallel (Euc. 33, I. Ch. 32, I.). Therefore, the distances on the parallels to RT, through 1 and 3, etc., intercepted between any two consecutive lines, are equal to the distances on the same parallels between any other consecutive two (Euc. 34, I. Ch. 30, I.), and each of these distances is equal to $\frac{1}{10}$ of an inch.

Again, in the triangle REF, since ab is parallel to

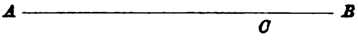
EF, we have, from similar triangles, $\frac{ab}{EF} = \frac{Ra}{RE} = \frac{1}{10}$; therefore $ab = \frac{EF}{10}$; but $EF = \frac{1}{10}$ of an inch; therefore $ab = \frac{1}{10}$ of $\frac{1}{10}$ of an inch, or $\frac{1}{100}$ of an inch.

In a similar manner it can be proved that CD equals $\frac{2}{100}$ of an inch, OP equals $\frac{8}{100}$ of an inch, etc.

If, on the left of the figure, we take UV equal to $\frac{1}{2}$ inch and divide it into ten equal parts, and divide $W4$ into ten equal parts, and draw a line from V to the first point M of $W4$, another line from the first point of UV to the second point of $W4$, and so on, we can prove gh equal to $\frac{1}{100}$ of $\frac{1}{2}$ inch or $\frac{1}{200}$ of an inch, and rs equal to $\frac{2}{100}$ of $\frac{1}{2}$ inch or $\frac{2}{200}$ of an inch, etc.

Suppose, now, we are required to represent a straight line of 125 feet, on the scale of 100 feet to the inch.

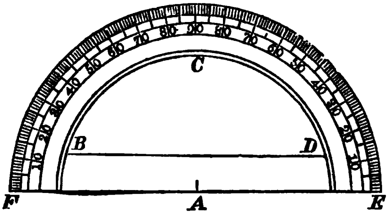
We draw any line AB; then placing one point of the dividers at O, on the plane-scale, and bringing the other to the point A, the intersection of the two lines 5 5 and $2L$, we lay off on AB a line AC equal to OA. Then AC is the required line, as it is equal to 1 inch and $\frac{25}{100}$ inch, or to $\frac{125}{100}$ of an inch.



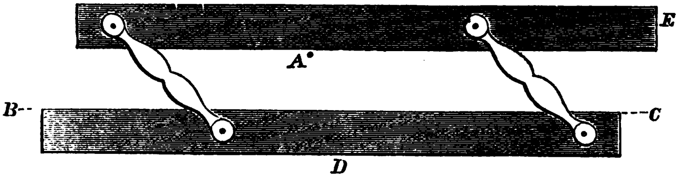
The two instruments described above are all that are *necessary* for drawing lines of a given length, or laying down angles of a given number of degrees. The following, however, are very convenient instruments:

A *protractor*, for laying down angles, and a *parallel* ruler for drawing parallel lines.

A *protractor* is a semicircle, usually of brass or some other metal, a part of the semicircle being cut out, as the part *B C D*. The outer rim left is divided into degrees, and sometimes into half degrees or even smaller parts, the lines of division all converging to a notch (*A*) on the diameter of the semicircle. The degrees are numbered from both left to right and right to left, from 10° to 170°, or from 10° to 90°, as in the figure.



At a given point in a given line, to lay down an angle of a given number of degrees by means of the protractor, place the notch, *A*, at the given point, and the line *F E* coincident with the given line. Then put a dot, or point, opposite the given number of degrees, on the outer or inner edge of the protractor. A line, drawn from this point to the given point, will make with the given straight line the required angle.



A *parallel rule* consists of two rules joined together by two pieces of metal of equal length, at equal distances apart.

To draw a line through a given point parallel to a given line, place one edge of one of the rulers coincident with the given line, and one edge of the other ruler on the point, and draw the line through the given point. It will be parallel to the given line.

Thus, suppose *A* to be the given point and *B C* the given straight line. Make the edge of the ruler, *D*, coincide with *B C*, and place the edge of the rule, *E*, on the point *A*. A line drawn through *A* will be parallel to *B C*.

The parallel line might also have been drawn by placing the upper edge of *E* on the point, or by placing the lower edge of *D* coincident with the given line *B C*.

EXAMPLE 1. Draw the right-angled triangle of which the hypotenuse is 236 feet, and one acute angle is 30°, and find by measurement the base and perpendicular. *Ans.* Perpendicular = 118; base = 204, nearly.

2. Draw the triangle whose two sides are, respectively, 120 and 80, and the included angle 42°, and find by measurement the third side.

***Ans.* 81, almost.**

3. Draw the right-angled triangle whose sides about the right angle are each 345, and find by measurement the hypotenuse.

4. Draw the triangle of which one side is 421, and the adjacent angles are 35° and 72°, and find by measurement the other sides.

CHAPTER II.

PLANE TRIGONOMETRY.—DEFINITIONS.

ARTICLE 1. TRIGONOMETRY is that branch of mathematics which treats of *angles*, *the relations of different angles to each other*, *and the relations of angles to lines*, *surfaces*, *and solids.*

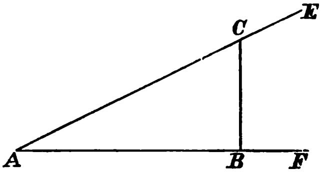
2. In PLANE TRIGONOMETRY the *angles* considered are such as are contained by *straight lines*, and the *surfaces* considered are *plane* surfaces.

3. The TRIGONOMETRICAL RATIOS of an *angle are ratios between the sides of a right-angled triangle*, *in which*, *or with respect to which*, *the angle has a certain position.*

The *right-angled triangle* is called *the triangle of reference.* It is formed by *the lines containing the angle* and a *perpendicular* to one of the lines, or to one of the lines produced, drawn from any point in the other line.

The trigonometrical ratios are six in number.

4. We will first consider the *trigonometrical ratios* of an *acute angle.*



Let A be any acute angle, formed by the two lines $A\,E$ and $A\,F$, meeting at the point A.

In $A\,E$ take any point

C, and from C draw $C\,B$ at right angles to $A\,F$. Then there is formed the right-angled triangle $A\,B\,C$, the *triangle of reference.* In this triangle, calling $C\,B$, the *side opposite* the given angle, the *perpendicular*, and $A\,B$, the *side adjacent* to the given angle, the *base*—

(1.) The Sine of an angle is the ratio of the *perpendicular* to the *hypotenuse.*

Thus, Sine of A (written sin. A) $= \frac{B\,C}{A\,C}$.

(2.) The Tangent of an angle is the ratio of the *perpendicular* to the *base.*

Thus, Tangent of A (written tan. A) $= \frac{B\,C}{A\,B}$.

(3.) The Secant of an angle is the ratio of the *hypotenuse* to the *base.*

Thus, Secant of A (written sec. A) $= \frac{A\,C}{A\,B}$.

(4.) The Cosine of an angle is the ratio of the *base* to the *hypotenuse.*

Thus, Cosine of A (written cos. A) $= \frac{A\,B}{A\,C}$.

(5.) The Cotangent of an angle is the ratio of the *base* to the *perpendicular.*

Thus, Cotangent of A (written cot. A, or cotan. A) $= \frac{A\,B}{B\,C}$.

(6.) The Cosecant of an angle is the ratio of the *hypotenuse* to the *perpendicular.*

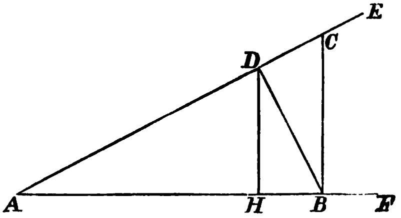
Thus, Cosecant of A (written cosec. A) $= \frac{A\,C}{B\,C}$.

5. If we take C as the acute angle, $A\,B$ as the per-

pendicular, and BC as the base, we shall have sin. $C = \frac{AB}{AC}$, tan. $C = \frac{AB}{BC}$, and sec. $C = \frac{AC}{BC}$. Now, C is the complement of A, as their sum equals 90°. Comparing the cos. A with the sin. C, it will be seen that they are identical, both being $\frac{AB}{AC}$. Also, tan. C is identical with cotan. A, and sec. C with cosec. A. The *cosine* of an angle is, therefore, the *sine* of its *complement;* the *cotangent* of an angle the *tangent* of its *complement;* and the *cosecant* of an angle the *secant* of its *complement;* so that the six trigonometrical ratios of an acute angle are really composed of three ratios belonging to that angle, and three ratios belonging to its complement.

6. The TRIGONOMETRICAL RATIOS, *for each acute angle, are* CONSTANT; i. e., *are the same for the same* or equal angle, *in whatever right-angled triangle it is situated.*

Let A represent any acute angle formed by the two lines AE and AF, meeting at A. Make ABC the triangle of reference. From B draw BD perpendicular to AC, and from D, where BD meets AC, draw a perpendicular DH to AB.



In triangle ADH, sin. $A = \frac{DH}{AD}$; in triangle ABC, sin. $A = \frac{BC}{AC}$; and in triangle $A\,B\,D$, sin. $A = \frac{B\,D}{A\,B}$ ((1) Art. 4). But $\frac{D\,H}{A\,D} = \frac{B\,C}{A\,C}$(Euc. 4, VI. Ch. 4, III.) $= \frac{B\,D}{A\,B}$ (Euc. 8, VI. Ch. 13, III.).

The angle $A = D\,B\,C$ (Euc. 8, VI. Ch. 13, III.) $= BDH$ (Euc. 29, I. Ch. 13, I.). Sin. $DBC = \frac{D\,C}{B\,C} = \frac{B\,C}{A\,C}$ $=$ sin. A. Also, sin. $B\,D\,H = \frac{HB}{BD} = \frac{B\,D}{A\,B} = \frac{B\,C}{A\,C}$ $=$ sin. A.

In triangle $A\,D\,H$, tan. $A = \frac{D\,H}{A\,H}$; in triangle $A\,B\,C$, tan. $A = \frac{B\,C}{AB}$; and in triangle $A\,B\,D$, tan. $A = \frac{B\,D}{AD}$. But $\frac{D\,H}{A\,H} = \frac{B\,C}{AB} = \frac{B\,D}{A\,D}$.

Also, tan. $D\,B\,C = \frac{D\,C}{DB} = \frac{B\,D}{A\,D} =$ tan. A; tan. $H\,D\,B = \frac{HB}{HD} = \frac{B\,D}{A\,D} = \frac{HD}{A\,H} =$ tan. A.

In triangle $A\,D\,H$, sec. $A = \frac{A\,D}{A\,H}$; in $A\,B\,C$, sec. A $= \frac{A\,C}{A\,B}$; and, in $A\,B\,D$, sec. $A = \frac{A\,B}{A\,D}$. But $\frac{A\,D}{A\,H}$ $= \frac{A\,C}{A\,B} = \frac{A\,B}{A\,D}$.

Also, sec. $D\,B\,C = \frac{B\,C}{B\,D} = \frac{A\,C}{A\,B} =$ sec. A; and sec. $H\,D\,B = \frac{B\,D}{H\,D} = \frac{A\,B}{A\,D} =$ sec. A.

In a similar manner it may be proved that the ratios for the cosine, cotangent, and cosecant of the same or equal angle are constant.

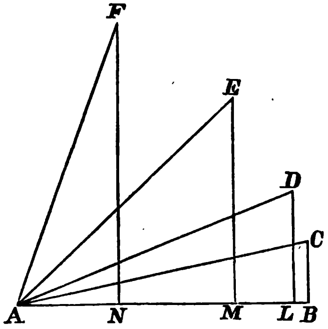
7. The Sine *and the* Cosine, *of an acute angle, are always* less than 1.

The *sine* being equal to the ratio of the perpendicular to the hypotenuse of the triangle of reference, in the fraction expressing that ratio the numerator is always less than the denominator, and therefore the value of the fraction is less than 1.

The *cosine* being equal to the ratio of the base to the hypotenuse, in the fraction expressing that ratio the numerator is always less than the denominator, and therefore the value of the fraction is less than 1.

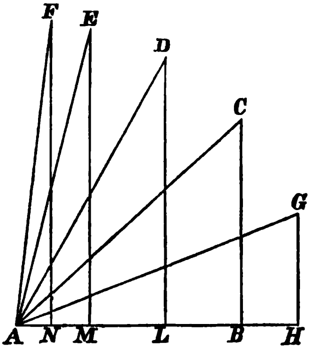
Thus let ABC, ALD, AME, and ANF, be a series of right-angled triangles, all having an hypotenuse of the same length. Let the hypotenuse of each triangle represent 32 units of length. Let $CB = 6$ units; $DL = 12$ units; $ME = 22$ units; $NF = 31$ units. The numerical measures of these lines are, therefore, 32, 6, 12, 22, etc.

Sin. $CAB = \frac{CB}{AC}$ =((1) Art. 4). $= \frac{6}{32}$ (Ch. Art. 43, II.); sin. $DAL = \frac{DL}{DA} = \frac{12}{32}$; sin. $EAM = \frac{EM}{AE} = \frac{22}{32}$; sin. $FAN = \frac{FN}{FA} = \frac{31}{32}$; all less than 1.



Also, let there be a series of right-angled triangles GAH, ABC, etc., all having an hypotenuse of the same length, viz., 16. Let $AH = 15$, $AB = 12$, $AL = 8$, $AM = 4$, and $AN = 2$, in terms of the unit of length.

Cos. $GAH = \frac{AH}{AG}$ ((4) Art. 4) $= \frac{15}{16}$; cos. $CAB = \frac{AB}{AC} = \frac{12}{16}$; cos. $DAL = \frac{AL}{AD} = \frac{8}{16}$; etc., all less than 1.



Art. 8. Cor. As the *acute* angle increases, the sine increases, but the cosine decreases.

9. (Defs.) If the cosine of an angle be subtracted from unity, the remainder is called the VERSED SINE of the angle. Thus 1 — cos. A = versin. A.

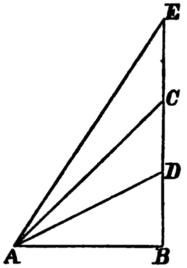
If the sine of an angle be subtracted from unity, the remainder is called the COVERSED SINE of the angle. Thus 1 — sin. A = coversin. A.

10. The TANGENT and the COTANGENT, of *an acute angle, may be* LESS THAN 1, EQUAL TO 1, or GREATER THAN 1.

The tangent being equal to the ratio of the perpendicular to the base of a right-angled triangle, the fraction expressing the ratio will be less than 1, when the perpendicular is less than the base; will be equal to 1, when the perpendicular is equal to the base; will be greater than 1, when the perpendicular is greater than the base.

The tangent of an angle will be less than 1, when the angle is less than 45°; will be equal to 1, when the angle is equal to 45°; will be greater than 1, when the angle is greater than 45°.

Thus let *ABD*, *ABC*, and *ABE*, be triangles right angled at *B*. Let the numerical measure of *AB* and of *BC* each equal 10; let the numerical measure of *BD* equal 5, and of *BE* equal 15; all referred to the same unit of length.



ABC is an isosceles triangle, and the angles *BAC* and *BCA* are each equal to 45° (Euc. 5 and 32, I. Ch. 18 and 25, I.);

Therefore *DAB* is less than 45°, and *EAB* is greater than 45°.

Now, $\tan. DAB = \frac{DB}{AB}$((2) Art. 4)$= \frac{5}{10} = \frac{1}{2}$;

Therefore the tangent of an angle less than 45° is less than 1.

$\text{Tan. } CAB = \frac{CB}{AB} = \frac{10}{10} = 1$;

Therefore the tangent of an angle of 45° equals 1.

$\text{Tan. } EAB = \frac{EB}{AB} = \frac{15}{10} = 1\frac{1}{2}$;

As above, the tangent of an angle greater than 45° is greater than 1.

Again, the cotangent of an angle being equal to the ratio of the base to the perpendicular of a right-angled triangle, is less than 1, equal to 1, or greater than 1, according as the base is less than the perpendicular, equal to the perpendicular, or greater than the perpendicular; that is, according as the angle is greater than 45°, equal to 45°, or less than 45°.

Taking the construction of the above figure,

$\text{Cot. } EAB = \frac{AB}{BE}$((5) Art. 4)$= \frac{10}{15} = \frac{2}{3}$;

Therefore the cotangent of an angle greater than 45° is less than 1.

$\text{Cot. } CAB = \frac{AB}{BC} = \frac{10}{10} = 1$;

Therefore the cotangent of 45° equals 1.

$$\text{Cot. } DAB = \frac{AB}{BD} = \frac{10}{5} = 2;$$

Therefore the cotangent of an angle less than 45° is greater than 1.

11. Cor. As the *acute* angle increases, its *tangent increases*, but its *cotangent decreases.*

12. The SECANT *and the* COSECANT, *of an acute angle, are* ALWAYS GREATER THAN 1.

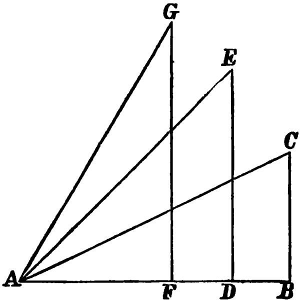
The secant of an acute angle being equal to the ratio of the hypotenuse to the base of a right-angled triangle, the fraction expressing that ratio always has its numerator greater than its denominator, and is therefore always greater than 1.

Let ABC, ADE, and AFG, be triangles right angled at B, D, and F, respectively. Let the numerical measure of AC, AE, and AG, each equal 20; let the numerical measure of AB equal 18, of AD equal 14, and of AF equal 10.

$$\text{Sec. } CAB = \frac{AC}{AB} = \frac{20}{18} = \frac{10}{9} = 1\tfrac{1}{9};$$

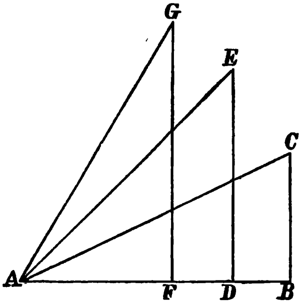
$$\text{Sec. } EAD = \frac{AE}{AD} = \frac{20}{14} = 1\tfrac{3}{7};$$

$$\text{Sec. } GAF = \frac{AG}{AF} = \frac{20}{10} = 2.$$



Each of these secants is greater than 1, and, as above, the secant of an acute angle is always greater than 1.

Again, the cosecant being equal to the ratio of the hypotenuse to the perpendicular, the fraction expressing that ratio is always greater than 1, as its numerator is always greater than its denominator.



Let ABC, ADE, and AFG, be triangles right angled at B, D, and F, respectively. Let the numerical measures of AC, AE, and AG, each equal 20; let the numerical measure of BC equal 8, of DE equal 14, and of FG 16, respectively.

$$\text{Cosec. } CAB = \frac{AC}{CB} = \frac{20}{8} = 2\tfrac{1}{2};$$

$$\text{Cosec. } EAB = \frac{AE}{DE} = \frac{20}{14} = 1\tfrac{3}{7};$$

$$\text{Cosec. } GAF = \frac{AG}{FG} = \frac{20}{16} = 1\tfrac{1}{4}.$$

Each of these is greater than 1, and, therefore, the cosecant of an acute angle is always greater than 1.

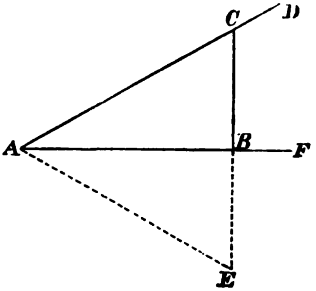
13. Cor. As the *acute* angle increases, its *secant increases*, but its *cosecant decreases*.

CHAPTER III.

TRIGONOMETRICAL RATIOS OF AN ANGLE OF 30°, OF 45°, AND OF 60°.

Art. 14. *To find the numerical values of the* TRIGONOMETRICAL RATIOS *of an angle of* 30°.

Suppose DAF to be an angle of 30°. In AD take any point, C, and from C draw CB perpendicular to AF, forming the triangle of reference, CAB. At the point A, in the line AB, make the angle BAE equal to BAC—that is, to 30°—and produce CB to meet AE at E. Then ABC and ABE are equal triangles (Euc. 26, I. Ch. 21, I.), and CB is equal to BE.



Now, in the triangle ACE, the angle A is equal to 60°, and the angle E is also equal to 60°. Therefore CA is equal to CE (Euc. 6, I. Ch. 27, I.), and CB, which is half of CE, is equal to half of CA.

Now, $AC^2 = AB^2 + BC^2$ (Euc. 47, I. Ch. 14, III.) $= AB^2 + \frac{1}{4}AC^2$;

Therefore $AB^2 = AC^2 - \frac{1}{4}AC^2 = \frac{3}{4}AC^2$; and $AB = \frac{\sqrt{3}}{2}AC$.

Now, $\sin. 30° = \sin. CAB = \frac{CB}{AC} = \frac{CB}{2CB} = \frac{1}{2}$;

$$\text{Tan. } 30° = \tan. CAB = \frac{CB}{AB} = \frac{\frac{1}{2}AC}{\frac{\sqrt{3}}{2}AC} = \frac{1}{\sqrt{3}}$$
$$= \tfrac{1}{3}\sqrt{3};$$

$$\text{Sec. } 30° = \sec. CAB = \frac{AC}{AB} = \frac{AC}{\frac{\sqrt{3}}{2}AC} = \frac{2}{\sqrt{3}}$$
$$= \tfrac{2}{3}\sqrt{3};$$

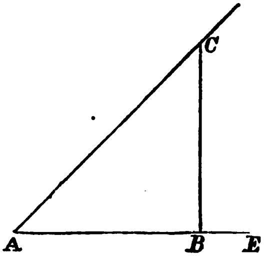
$$\text{Cos. } 30° = \cos. CAB = \frac{AB}{AC} = \frac{\frac{\sqrt{3}}{2}AC}{AC} = \tfrac{1}{2}\sqrt{3};$$

$$\text{Cot. } 30° = \cot. CAB = \frac{AB}{BC} = \frac{\frac{\sqrt{3}}{2}AC}{\frac{1}{2}AC} = \sqrt{3};$$

$$\text{Cosec. } 30° = \text{cosec. } CAB = \frac{AC}{BC} = \frac{AC}{\frac{1}{2}AC} = 2;$$

Versin. $30° = \text{versin. } CAB = 1 - \cos. 30°$ (Art. 9)
$$= 1 - \frac{\sqrt{3}}{2}.$$

15. *To find the numerical values of the* TRIGONOMETRICAL RATIOS *of an angle of* 45°.



Suppose CAE to be an angle of 45°. In AC take any point, C, and from C draw CB perpendicular to AE, meeting AE at B.

In the right-angled triangle ABC, as the angle A is equal to 45°, the angle C is also equal to 45° (Euc. 32, I. Ch. 18, I.). Consequently C is equal to A, and the side AB to BC (Euc. 6, I. Ch. 27, I.).

Now, $AC^2 = AB^2 + BC^2 = 2\overline{BC}^2 = 2\overline{AB}^2$;

therefore $AB = \dfrac{AC}{\sqrt{2}} = BC$.

Now, $\sin. 45° = \sin. A = \dfrac{BC}{AC} = \dfrac{\frac{1}{\sqrt{2}} AC}{AC} = \dfrac{1}{\sqrt{2}}$ $= \frac{1}{2}\sqrt{2}$;

$$\text{Tan. } 45° = \tan. A = \frac{BC}{AB} = \frac{BC}{BC} = 1;$$

$$\text{Sec. } 45° = \sec. A = \frac{AC}{AB} = \frac{AC}{\frac{1}{\sqrt{2}} AC} = \sqrt{2};$$

$$\text{Cos. } 45° = \cos. A = \frac{AB}{AC} = \frac{\frac{1}{\sqrt{2}} AC}{AC} = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2};$$

$$\text{Cot. } 45° = \cot. A = \frac{AB}{BC} = \frac{AB}{AB} = 1;$$

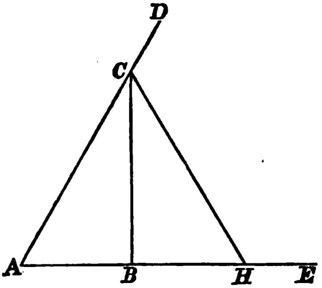
$$\text{Cosec. } 45° = \text{cosec. } A = \frac{AC}{BC} = \frac{AC}{\frac{1}{\sqrt{2}} AC} = \sqrt{2};$$

$$\text{Versin. } 45° = \text{versin. } A = 1 - \frac{1}{\sqrt{2}} = 1 - \tfrac{1}{2}\sqrt{2}.$$

16. *To find the numerical values of the* TRIGONOMETRICAL RATIOS *of an angle of* 60°.

Suppose DAE to be an angle of 60°. In AD take any point, C, and from C draw CB perpendicular to AE, meeting AE in B. From B lay off BH, equal to AB, and join CH.

Then the triangles CAB and BCH are equal (Euc. 4, I. Ch. 20, I.), and the angle CHA is equal to CAH—that is, to 60°. Therefore the angle ACH is also equal to 60°. In the triangle ACH the side AH is equal to AC, and AB, which is one-half of AH, is equal to one-half of AC.



Now, $AC^2 = AB^2 + BC^2$; therefore $BC^2 = AC^2 - AB^2 = AC^2 - \frac{1}{4}AC^2 = \frac{3}{4}AC^2$ and $BC = \frac{\sqrt{3}}{2}AC$.

$$\text{Now, sin. } 60^\circ = \text{sin. } A = \frac{BC}{AC} = \frac{\frac{\sqrt{3}}{2}AC}{AC} = \tfrac{1}{2}\sqrt{3};$$

$$\text{Tan. } 60^\circ = \text{tan. } A = \frac{BC}{AB} = \frac{\frac{\sqrt{3}}{2}AC}{\frac{1}{2}AC} = \sqrt{3};$$

$$\text{Sec. } 60^\circ = \text{sec. } A = \frac{AC}{AB} = \frac{AC}{\frac{1}{2}AC} = 2;$$

$$\text{Cos. } 60^\circ = \text{cos. } A = \frac{AB}{AC} = \frac{\frac{1}{2}AC}{AC} = \frac{1}{2};$$

$$\text{Cot. } 60^\circ = \text{cot. } A = \frac{AB}{BC} = \frac{\frac{1}{2}AC}{\frac{\sqrt{3}}{2}AC} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3};$$

$$\text{Cosec. } 60^\circ = \text{cosec. } A = \frac{AC}{BC} = \frac{AC}{\frac{\sqrt{3}}{2}AC} = \frac{2}{\sqrt{3}} =$$

$\frac{2}{3}\sqrt{3}$;

$$\text{Versin. } 60^\circ = \text{versin. } A = 1 - \tfrac{1}{2} = \frac{1}{2}.$$

We might also derive the trigonometrical ratios of an angle of 60° from those of an angle of 30°, according to Art. 5.

Thus, sin. 60° = cos. 30° = $\frac{1}{2}\sqrt{3}$;
Tan. 60° = cot. 30° = $\sqrt{3}$;
Sec. 60° = cosec. 30° = 2;
Cos. 60° = sin. 30° = $\frac{1}{2}$;
Cot. 60° = tan. 30° = $\frac{1}{3}\sqrt{3}$;
Cosec. 60° = sec. 30° = $\frac{2}{3}\sqrt{3}$;
Versin. 60° = 1 − cos. 60° = 1 − sin. 30° = $\frac{1}{2}$.

EXAMPLE 1. If the hypotenuse of a right-angled triangle be 5, and the perpendicular be 4, required the trigonometrical ratios for the angle at the base. Required also the trigonometrical ratios for the angle at the perpendicular.

2. If the base and perpendicular of a right-angled triangle be 7 and 8, find the trigonometrical ratios of the angles at the base and perpendicular.

3. Calculate to four decimal places the numerical values of the trigonometrical ratios of an angle of 30°, and of 60°.

4. Calculate to four decimal places the numerical values of the trigonometrical ratios of an angle of 45°.

CHAPTER IV.

THE USE OF TRIGONOMETRICAL TABLES.—SOLUTION OF RIGHT-ANGLED TRIANGLES.

ART. 17. The *trigonometric ratios* for all angles between 0° and 90°, beginning at an angle of 1′ and increasing in size by successive additions of 1′, have been calculated and have been arranged in tables. Tables have also been calculated for angles beginning at an angle of 10″, and increasing in size by successive additions of 10″. Smaller tables, in which the interval between angles is 15′, have also been arranged for calculations where great accuracy is not required.

Such tables are called in general *Trigonometrical Tables*, and also "Tables of Natural Sines and Cosines," "Tables of Natural Tangents and Cotangents," etc.

18. The *logarithmic* values of the trigonometric ratios have also been arranged in tables, called "Tables of Logarithmic Sines," etc., for use in calculations by logarithms.

19. The values of ratios, intermediate between the ratios of the tables, are obtained from those of the tables on the theory that "*for small intervals, the differences of the ratios are proportional to the differences of the angles.*"

Thus suppose we are using a table of natural sines, in which the sines are calculated for intervals of 1′, and we are required to find the sine of an angle of 10° 1′ 1″. This sine falls between the sine of 10° 1′ and the sine of 10° 2′. In the table of sines, under 10° and opposite 1 in the column of Min. (minutes), we find the sine of 10° 1′ to be .173935, and, directly under it, and opposite 2 in the column headed Min., we find the sine of 10° 2′ to be .174221. (The decimal point is not prefixed in the table, but is always understood.) The difference of the two sines is $\frac{286}{1,000,000}$, or .000286, and the angles differ by 1′ or 60″. Therefore since when the angle increases 60″ the sine increases .000286, on the principle just enunciated, when the *angle* increases 1″, or $\frac{1}{60}$th of the former increase, the *sine* will also increase $\frac{1}{60}$th of its former increase, or will increase $\frac{.000286}{60} = \frac{47}{10,000,000} = .000005$, nearly. So that the sine of 10° 1′ 1″ is obtained by adding .000005 to .173935. The sine of 10° 1′ 1″ is therefore .173940.

If we wished to obtain the sine of 10° 1′ 2″ we should add twice the increase for one second, or $\frac{2}{60}$ths of the increase for 1′—that is, .000009, nearly—and so on.

The increase for 1″ is generally calculated at short intervals, and put in the table under the column of the ratios to which it belongs, with the name of "*proportional parts*." To find the increase for any number of seconds, we multiply the proportional parts for 1″ by the number of seconds.

To obtain from the tables a ratio for an angle between two angles of the table, it is best in general to take the ratio belonging to the *smaller* of the two angles, between which it falls, and apply the correction according to the principle above given, or directly from the "Corrections" or "Proportional Parts" of such tables, being careful to *add such correction, according as the ratio required is a sine, tangent, or secant;* and to *subtract the correction, if the required ratio is a cosine, cotangent, or cosecant* (Arts. 8, 11, 13).

20. Conversely, *to obtain the degrees, minutes, and seconds, answering to a given ratio intermediate between two ratios of the table, take the degrees and minutes belonging to the smaller of the two angles,* between whose ratios it falls, *and, for the seconds, divide the difference between this ratio and the given ratio by the proportional parts for* 1″, *found under the column in which the two ratios appear.*

Thus suppose we are using a five-place table, and we have a sine given as .50060, and are required to find the number of degrees, minutes, and seconds, in the angle to which it belongs.

Looking in the table of sines we find it falls between .50050 and .50076; that is, that it belongs to an angle between 30° 2′ and 30° 3′. The difference between the sine belonging to the smaller angle and the given sine is 10—that is, $\frac{10}{100,000}$—and the proportional part for 1″ is $\frac{26}{60} = 0.43$ (considering 26 as a whole number). Dividing 10 by .43 we have for the seconds 23. Therefore the angle whose sine is .50060, is an angle of 30° 2′ 23″.

21. The logarithm of a ratio intermediate between the logarithms of two ratios is obtained on the same general principle as were the ratios themselves, it being assumed that, for small intervals, the difference of the logarithms of the ratios is proportional to the difference of the angles.

The principle here assumed is not strictly true, nor is the principle assumed in the previous article strictly true; but, employing it with the limitation *of small intervals,* we are not in danger of error, except in certain angles near 0°, and near 90°, for which angles separate tables are provided.

22. In the tables of logarithmic sines, cosines, tan-

gents, etc., the logarithms are generally calculated for intervals of ten seconds, and the corrections or proportional parts are given for the seconds from 1″ to 9″.

Most of the trigonometric ratios, whose logarithms are given, are less than 1 (Arts. 7 and 10), and, therefore, the characteristics of their logarithms would be negative (Art. 404, Loomis's Alg.). To avoid the use of negative characteristics, however, 10 is always added to the logarithm. Consequently, in order to obtain the correct logarithm of a result, *in a calculation in which trigonometric ratios have been represented by their logarithms*, from the resulting logarithm *10 must be subtracted for every trigonometric ratio used as a multiplier*, and, to the resulting logarithm, *10 must be added for every trigonometric ratio used as a divisor.*

SOLUTION OF RIGHT-ANGLED TRIANGLES.

Art. 23. The *parts* of any triangle are the sides and the angles.

To *solve* a triangle is to find the unknown parts from certain known parts.

***Trigonometry* was, primarily, the science by which triangles are solved, and was originally so defined.**

24. The parts of *any* triangle are six in number. In a *right-angled* triangle, as one of the parts is always a right angle, *one* part is always known. As the two acute angles together make a right-angle, when *one* acute angle is known, the *other*, being its complement, is also known. So that to *solve* a *right-angled* triangle it is only necessary to consider *four parts;* viz., the

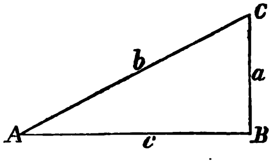
three sides and one acute angle. Any two of these parts of a right-angled triangle being known, we are able, by the use of trigonometric tables, to solve the triangle.

25. Suppose two sides are known. These may be the *hypotenuse* and *perpendicular*; the *hypotenuse* and *base*; or the *perpendicular* and *base*.

As the perpendicular is "the side opposite the given angle" (Art. 4), either of the sides about the right angle may be made the perpendicular, and the other side the base, or "the side adjacent to the given angle," according as we take one or the other of the acute angles as the given angle. When two sides are given we really have, therefore, two cases only, viz.: 1. When the hypotenuse and a side are given; 2. When the two sides about the right angle are given.

26. *The hypotenuse and a side being known, to solve the triangle.*

Let ABC be a right-angled triangle, having its right angle at B. Denote the sides opposite the angle by small letters of the same name as the capital letters denoting the angles.



(Generally, in all triangles, the notation will be adopted of capitals for the angles, and small letters of the same name for the opposite sides.)

In the triangle ABC suppose we have given the hypotenuse b, and the side a.

(1) Sin. $A = \frac{a}{b}$ ((1) Art. 4).

(2) Also $\frac{c}{b} =$ cos. A ((4) Art. 4); or, $c = b$ cos. A.

(3) c also equals $\sqrt{b^2 - a^2} = \sqrt{(b+a)(b-a)}$; a formula convenient for the use of logarithms, as then $\log. c = \frac{1}{2}\{\log. (b+a) + \log. (b-a).\}$

From (1) we see that, when the hypotenuse and a side are given, *the angle opposite the given side is an angle whose sine is equal to the ratio of the side to the hypotenuse.* It is, therefore, to be found from the table of sines.

When this angle is found, the third side can be found by formula (2).

27. If the hypotenuse, b, and the side, c, were given, we could find the angle C, as we found the angle A; or, if we desired to find the angle A, we have

$$\frac{c}{b} = \cos. A;$$

that is, when the hypotenuse and a side are given, the angle *adjacent to the given side is an angle whose cosine is the ratio of the given side to the hypotenuse.* It is, therefore, to be found from a table of cosines.

In this case the other side, $a = b \sin. A$;

$$\text{for } \frac{a}{b} = \sin. A; \therefore a = b \sin. A;$$

$$\text{or, } a = \sqrt{b^2 - c^2} = \sqrt{(b+c)(b-c)}.$$

EXAMPLE 1. If the hypotenuse and perpendicular of a right-angled triangle be 192 and 130, respectively, required the other parts of the triangle. *Ans.* Angles = 47° 23′ 2″ and 42° 36′ 58″; side = 141.29.

2. The hypotenuse and base of a right-angled triangle being 71 and 64, respectively, required the other parts of the triangle.

Ans. Angles = 25° 39′ 22″ and 64° 20′ 38″; side = 30.74.

3. If the hypotenuse of a right-angled triangle be 71, and the base 60, required the other parts.

Ans. Angles = 32° 19′ 14″ and 57° 40′ 46″; side = 37.96.

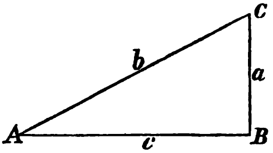
4. If the hypotenuse and a side of a right-angled triangle be 140 and 84, respectively, what are the other parts?

5. If the hypotenuse and a side of a right-angled triangle be 130 and 66, respectively, what are the other parts?

6. The hypotenuse and perpendicular of a right-angled triangle are 200 and 100, respectively; what are the other parts?

Art. 28. *The two sides about the right angle being known, to solve the triangle.*

In the figure, suppose the two sides, a and c, are known, and it is required to find the hypotenuse and angles.



$b = \sqrt{a^2 + c^2}$ (Euc. 47, I. Ch. 14, III.).

(1) Tan. $A = \frac{a}{c}$ ((2) Art. 4).

After the angle A is found, we can also find the hypotenuse, thus:

(2) $\frac{a}{b} = \sin. A$; $\therefore b = \frac{a}{\sin. A}$; or,

(3) $\frac{b}{c} = \sec. A$; $\therefore b = c \sec. A$.

From (1) we see that *when the two sides of a right-angled triangle are given, the angle opposite either of the sides is an angle whose tangent is the ratio of that side to the other side.* It is, therefore, to be found from a table of tangents.

Example 1. If the base of a right-angled triangle be 141, and the perpendicular be 193, required the angles and the hypotenuse.

Ans. 53° 50′ 57″, 36° 9′ 3″; 239.02, nearly.

2. If the two sides about the right angle of a right-angled triangle be 2.1 and 2, what are the angles and the hypotenuse?

Ans. 43° 36′ 10″, 46° 23′ 50″; 2.9.

3. If the two sides of a right-angled triangle be 123.5 and 10.16, what are the angles and the hypotenuse?

Ans. 4° 42′ 11″, 85° 17′ 49″; 123.92.

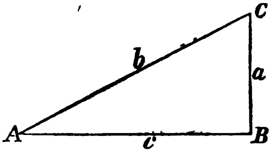
4. If the two sides of a right-angled triangle be 39 and 30, what are the other parts?

5. If the two sides of a right-angled triangle be 81.48 and 108.64, what are the other parts?

6. If the two sides of a right-angled triangle are 131 and 13.1, find the other parts.

7. If the sides of a right-angled triangle are each equal to a, what is the hypotenuse and what are the angles?

29. Now, suppose a side and an acute angle are known. These may be either the hypotenuse and an acute angle, or one of the sides about the right angle and an acute angle.



30. *The hypotenuse and an acute angle being known, to solve the triangle.*

In the right-angled triangle, $A\ B\ C$, suppose the hypotenuse b, and the angle A, are known.

$$(1)\quad \text{Sin. } A = \frac{a}{b}; \;\therefore\; a = b \text{ sin. } A.$$

$$(2)\quad \text{Cos. } A = \frac{c}{b}; \;\therefore\; c = b \text{ cos. } A.$$

In the same triangle suppose the hypotenuse b, and the angle C, are known.

$$(3)\quad \text{Sin. } C = \frac{c}{b}; \;\therefore\; c = b \text{ sin. } C.$$

$$(4)\quad \text{Cos. } C = \frac{a}{b}; \;\therefore\; a = b \text{ cos. } C.$$

From (1) and (3) it will be seen that, in a right-angled triangle, *the side opposite an acute angle is equal to the product of the hypotenuse by the sine of the angle.*

From (2) and (4) it will be seen that, in a right-angled triangle, *the side adjacent to an acute angle is equal to the product of the hypotenuse by the cosine of the angle.*

Example 1. If the hypotenuse of a right-angled triangle be 4.958, and one of the angles be 54° 44′, find the other sides.

Ans. 4.048, 2.8626.

2. The hypotenuse of a right-angled triangle is 25, and one of the acute angles is 16° 15′ 37″. Required the sides. *Ans.* 7 and 24.

3. The hypotenuse of a right-angled triangle is 37.36, and one of the acute angles is 12° 30′. Required the sides. *Ans.* 8.0862, 36.474.

4. The hypotenuse of a right-angled triangle is 120, and one of the acute angles is 36° 14′ 15″. Required the sides.

Ans. 70.936, 96.789.

5. The hypotenuse of a right-angled triangle is 316.5, and one of the acute angles is 55° 30′ 17″. Required the sides.

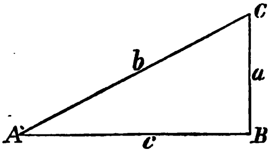
6. The hypotenuse of a right-angled triangle is 656.15, and one of the acute angles is 75° 10′ 5″. Required the sides.

7. The hypotenuse of a right-angled triangle is 100, and one of the acute angles is 60°. Required the sides.

8. The hypotenuse of a right-angled triangle is 567, and one of the acute angles is 45°. Required the sides.

9. The diagonal of a square is 400. Required the length of a side.

10. The side of a rhombus is 48 feet long, and one of its angles is 68°. Required the length of its diagonals.



31. *One of the sides about the right angle and one of the acute angles being known, to solve the triangle.*

In the right-angled triangle $A\,B\,C$, suppose the side a, and the angle A, opposite the side a, are known.

(1) $\text{Sin.}\,A=\frac{a}{b}$; $\therefore\ b=\frac{a}{\text{sin.}\,A}$.

(2) $\text{Tan.}\,A=\frac{a}{c}$; $\therefore\ c=\frac{a}{\text{tan.}\,A}$.

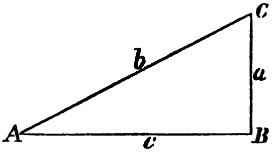
(3) Or $\text{cosec.}\,A=\frac{b}{a}$; $\therefore\ b=a\ \text{cosec.}\,A$.

(4) $\text{Cot.}\,A=\frac{c}{a}$; $\therefore\ c=a\ \text{cot.}\,A$.

Formula (1) and formula (2) are more convenient for general use than formula (3) and formula (4).

From (1) and (2) it will be seen that when a side about the right angle of a right-angled triangle, and an angle opposite it, are given, *the hypotenuse is equal to the quotient of the given side divided by the sine of the given angle, and the other side is equal to the quotient of the given side divided by the tangent of the given angle.*

32. If, instead of a side and the opposite angle being known, we know a side and the adjacent angle, we can find the opposite angle—as each acute angle is the complement of the other—and then we can solve the triangle, as in Art. 31. Or we can solve the triangle directly, thus:

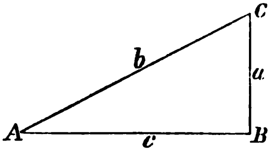


Suppose, in the right-angled triangle $A\,B\,C$, the side c and the angle A are given, to solve the triangle.

(1) $\text{Cos.}\,A=\frac{c}{b}$; $\therefore\ b=\frac{c}{\text{cos.}\,A}$.

(2) Tan. $A = \frac{a}{c}$; $\therefore$ $a = c$ tan. A.

(3) Also, sec. $A = \frac{b}{c}$; $\therefore$ $b = c$ sec. A.



From (2) and (3) it will be seen that, in a right-angled triangle, *the side opposite an acute angle is equal to the product of the side adjacent to that angle by the tangent of the angle;* and that *the hypotenuse is equal to the product of a side by the secant of the angle adjacent to the side.*

EXAMPLE 1. The side of a right-angled triangle is 141, and the angle opposite this side is 33° 41′ 6″. Required the hypotenuse and the other side. *Ans.* 254.225, 211.54.

2. The side of a right-angled triangle is 124.6, and the angle opposite it is 64° 20′. What is the hypotenuse and the other side? *Ans.* 138.24, 59.8766.

3. The side of a right-angled triangle is 19.67, and the angle opposite to it is 18° 31′ 4″. Required the hypotenuse and the other side.

4. The side of a right-angled triangle is 111.11, and the angle adjacent to this side is 78° 49′. Required the hypotenuse and the other side.

5. If one side of a rectangle is 50 feet, and the diagonal makes an angle of 30° with this side, what is the length of the diagonal, and what is the length of the other sides of the rectangle?

6. The diagonal of a rhombus is 30 feet, and the angle through which the diagonal passes is 120°. What is the length of a side of the rhombus, and what is the length of the other diagonal?

33. From the preceding articles it will be seen that to solve a right-angled triangle we simply apply the definitions of trigonometric ratios. To find a required part we select, in each case, the definition in which occur the required part and the two known parts.

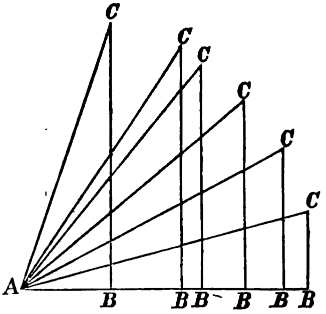
CHAPTER V.

TRIGONOMETRIC RATIOS OF A RIGHT ANGLE, OF 0, AND OF OBTUSE ANGLES.

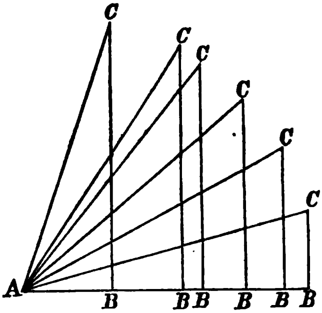
Art. 34. According to Art. 3, the *triangle of reference* for the trigonometric ratios of an angle is formed by dropping a perpendicular upon one side, or side produced containing the angle, from any point in the other side. In the case of a right angle, the perpendicular coincides with one of the sides of the angle and no triangle is formed. We shall, therefore, obtain the trigonometric ratios of a right angle by the method of limits, considering the case of a right-angled triangle, whose hypotenuse remains constant, while one of the acute angles, taken as a variable, "approaches indefinitely" to a right angle as its limit (Ch. Art. 28, Bk. V.).

Art. 35. *The sine of a right angle, or the sine of an angle of 90° is 1.*

Let $A B C$ be a triangle right-angled at B. Suppose, while the hypotenuse remains the same, the acute angle $B A C$ increases. At the same time $B C$ increases. Also the sine of $B A C$ increases (Art. 8).



Now, as the angle BAC increases, it approaches a right angle as its limit, and its sine approaches the sine of a right angle as its limit. Also as the angle BAC increases and approaches a right angle as its limit, the perpendicular, BC, approaches the hypotenuse, AC, as its limit (Euc. 6, I. Ch. 27, I.); and, therefore, $\frac{BC}{AC}$ approaches $\frac{AC}{AC}$, or 1, as its limit. But sin. BAC always equals $\frac{BC}{AC}$; therefore, at the limits, sin. 90°, or the sine of a right angle, equals 1 (Ch. Art. 29, Bk. V.).



36. *The cosine of a right angle, or the cosine of an angle of 90°, is 0.*

When the angle A (*see* figure) increases toward its limit (a right angle), its cosine decreases and approaches the cosine of a right angle as its limit (Art. 8). At the same time that A increases, AB decreases, and approaches 0 as its limit, so that the limit of $\frac{AB}{AC}$ is $\frac{0}{AC}$, or 0. But cos. A always equals $\frac{AB}{AC}$; therefore, at the limits, the cosine of a right angle, or cos. 90°, equals 0.

37. *The tangent of a right angle, or the tangent of an angle of 90°, equals* ∞.

When the angle A increases (*see* figure, page 40), the tangent increases (Art. 11). When the angle A approaches a right angle as its limit, its tangent approaches the tangent of a right angle as its limit. Also as the angle A increases, at the same time CB increases toward AC, as its limit, while AB decreases toward 0 as its limit, so that $\frac{BC}{AB}$ approaches $\frac{AC}{0}$ or ∞ as a limit. But tan. $A = \frac{BC}{AB}$; therefore, at the limits, the tangent of a right angle, or tangent of 90°, equals ∞.

38. By a similar method of proof, it can be shown that *the cotangent of a right angle is 0;* that *the secant of a right angle is* ∞; and that *the cosecant of a right angle is 1.*

39. *To find the trigonometric ratios of 0°.* As 0° is the complement of 90°, we can determine the trigonometric ratios of 0° from those of a right angle, according to Art. 5.

Thus sin. 0° = cos. 90° = 0;
Cos. 0° = sin. 90° = 1;
Tan. 0° = cot. 90° = 0;
Cot. 0° = tan. 90° = ∞;
Sec. 0° = cosec. 90° = 1;
Cosec. 0° = sec. 90° = ∞.

40. We can also find the trigonometric ratios of 0° by the method of limits, considering 0° as the limit to which an acute angle approaches indefinitely when decreasing.

Thus, take the case of the sine of 0°.

When A approaches 0° (*see* figure, page 40), as its limit, sin. A approaches sin. 0° as its limit, and CB approaches 0 as its limit. Therefore the limit of $\frac{CB}{AC}$ $=\frac{0}{AC}=0$. But sin. A always equals $\frac{CB}{AC}$. Therefore, at the limits, we shall have sin. 0° $=0$ (Ch. Bk. V., Art. 29).

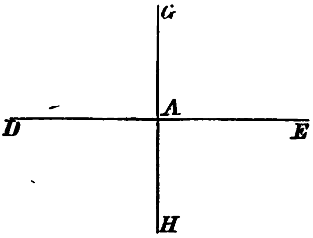
In a similar manner the other trigonometric ratios of 0° can be found.

TRIGONOMETRICAL RATIOS OF AN OBTUSE ANGLE.

Art. 41. Before proceeding to the trigonometrical ratios of an obtuse angle, we will define the *sign* of a line as distinguished from its *value*, or numerical measure.

The *sign* of a line generally denotes the *direction* in which it is measured. To lines measured in one direction there are given positive signs (not always prefixed), while to lines measured in the opposite direction there are given negative signs.

Thus, taking two lines, DE and GH (which we will name *initial lines*), intersecting at right angles at A, we call all lines going to the right of GH, or from A toward E, *positive*, and we call all lines to the left of GH, or going from A toward D, *negative.* So we call all perpendiculars upon DE, from above DE, *positive* lines, and we call all perpendiculars upon DE, from below, *negative.*

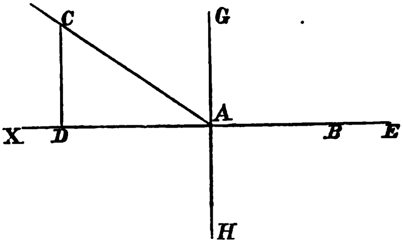


We shall also consider the hypotenuse of a right-angled triangle always positive.

Now, if we put the vertex of the angle, whose trigonometric ratios we are to consider, at A, and one of its sides coincident with $A\,E$, it will be apparent that (in the sense of our definitions with regard to the signs of lines), the *trigonometric ratios of all angles thus far considered are positive*, because the sides of the triangle of reference are all positive.

42. Take now an obtuse angle, and apply its vertex to the point A (see figure), and let one of its sides, as $A\,B$, be coincident with $A\,E$.

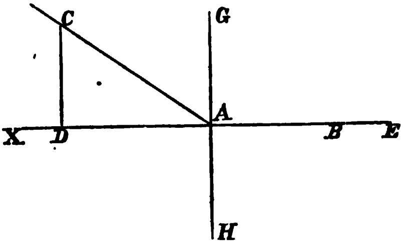
To construct the *triangle of reference* (Art. 3), we drop a perpendicular, $C\,D$, from any point, C, in $A\,C$, upon $A\,B$ produced, meeting $A\,B$ produced in D. In this triangle, $A\,D$, the side adjacent to the obtuse angle is negative, because we consider all lines running from A, toward the right of $G\,H$, as positive; and those from A, toward the left of $G\,H$, *negative* (Art. 41).



43. The trigonometrical ratios of an obtuse angle have the same *names* as the corresponding ratios of an acute angle, but have not always the same sign. For, as $A\,D$, the *base* of the triangle of reference, or the side *adjacent* to the obtuse angle, is always negative, and as the hypotenuse, $A\,C$, and the perpendicular,

CD, are always positive (Art. 41), every ratio in which AD occurs must be negative.

44. Thus (1), the *sine* of an *obtuse angle* is the ratio of the *perpendicular* to the *hypotenuse* of the triangle of reference, and is positive (Art. 41).



$$\text{Sin.}\,CAB = \frac{CD}{AC} = \text{positive quantity.}$$

(2.) The *tangent* of an *obtuse angle* is the ratio of the *perpendicular* to the *base*, and is *negative*.

$$\text{Tan.}\ CAB = \frac{CD}{AD} = \frac{\text{positive quantity}}{\text{negative quantity}} = \left\{\begin{array}{l}\text{negative}\\\text{quantity.}\end{array}\right.$$

(3.) The *secant* of an *obtuse angle* is the ratio of the *hypotenuse* to the *base*, and is *negative*.

$$\text{Sec.}\ CAB = \frac{AC}{AD} = \frac{\text{positive quantity}}{\text{negative quantity}} = \left\{\begin{array}{l}\text{negative}\\\text{quantity.}\end{array}\right.$$

(4.) The *cosine* of an *obtuse angle* is the ratio of the *base* to the *hypotenuse*, and is *negative*.

$$\text{Cos.}\ CAB = \frac{AD}{AC} = \text{negative quantity.}$$

(5.) The *cotangent* of an *obtuse angle* is the ratio of the *base* to the *perpendicular*, and is *negative*.

$$\text{Cot.}\ CAB = \frac{AD}{DC} = \text{negative quantity.}$$

(6.) The *cosecant* of an *obtuse angle* is the ratio of the *hypotenuse* to the *perpendicular* of the triangle of reference, and is *positive*.

Cosec. $CAB = \frac{AC}{CD} =$ positive quantity.

45. It can be proved that the *sine and cosine of an obtuse angle* are always *less than 1* in numerical value, in the same manner that the sine and cosine of an acute angle have been proved less than 1 (Art. 7); also, that *when an obtuse angle increases* its *sine decreases*, but that its *cosine increases* (negatively) in numerical value.

It may also be proved that *the tangent* and *cotangent of an obtuse angle may be equal to* -1, *greater than* -1, *or less than* -1, as it was proved that the tangent and cotangent of an acute angle might be equal to 1, greater than 1, or less than 1 (Art. 10); also, that as the *obtuse angle increases* its *tangent decreases*, but its *cotangent increases*, both *negatively*.

It may further be proved that the *secant* and *cosecant* of an obtuse angle are *always greater than 1* numerically (the *secant* being greater than -1, the cosecant being greater than $+1$), in the same manner that the secant and cosecant of an acute angle were proved to be always greater than 1 (Art. 12); also, that *as the obtuse angle increases*, the secant decreases (*negatively*), but its cosecant increases.

Def. The versed sine of an obtuse angle is equal to the *algebraic* difference between 1 and the cosine of the angle.

Thus (*see* figure, page 43), versin. $A = 1 - \cos. A$. But, as cos. A is a negative quantity (4, Art. 44), versin. A is greater than 1. (In the case of an acute angle, the versed sine is less than 1.)

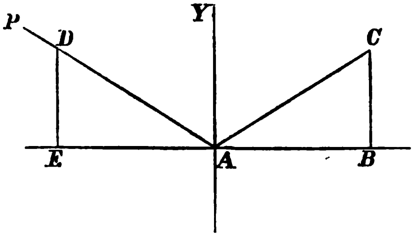
46. Any *trigonometrical ratio of an acute or obtuse*

angle has the same numerical value as the corresponding ratio of its supplement.

The SINE *and* COSECANT *of an acute angle, and the* SINE *and* COSECANT *of its supplementary obtuse angle, are all of the same sign, and are positive.*

The TANGENT, COTANGENT, COSINE, *and* SECANT *of an acute angle are positive, while the corresponding ratios of the supplementary obtuse angle are negative.*

In the figure CAB is an acute angle. Produce BA through A, beyond the perpendicular AY. At A, in the straight line AE, make the angle PAE equal to CAB. Then is PAB the supplement of PAE (Ch. Bk. I., Art. 19), and, consequently, equal to the supplement of CAB, the equal of DAE.



Construct the triangle of reference, CAB. In AP take AD equal to AC, and from D draw DE perpendicular to AE. Then is DAE, the triangle of reference for the obtuse angle DAB, equal to the triangle CAB (Euc. 26, I. Ch. 23, I.).

In the triangles ACB and ADE the hypotenuses AC and AD, and the perpendiculars CB and DE, and the base, AB, are all positive, while the base, AE, is negative (Arts. 41 and 42).

As the triangles of reference are equal in all respects, and therefore similar, the ratio between any two of the sides of one is equal to the ratio between the corresponding sides of the other (Euc. 4, VI. Ch. 4, III.); that is, the *numerical* value of any trigonometric ratio of the acute angle, $C A B$, is equal to the *numerical* value of the corresponding ratio of the obtuse angle, $D A B$.

Now, sin. $C A B = \frac{C B}{A C}$; also, sin. $D A B = \frac{D E}{D A}$. But $\frac{C B}{A C} = \frac{D E}{D A}$, as the triangles $C A B$ and $D A E$ are equal. Also as the lines $C B$, $D E$, $A C$, and $A D$, are all positive, the ratios $\frac{C B}{A C}$ and $\frac{D E}{D A}$ are positive, and therefore the sine of the angle, $C A B$, and the sine of its supplement, $D A B$, are of the same sign, and are positive.

Again, cosec. $C A B = \frac{A C}{C B}$; also, cosec. $D A B = \frac{D A}{D E}$. But $\frac{A C}{C B} = \frac{D A}{D E}$, and, as the terms of these ratios are *positive*, the ratios are positive; and, therefore, the cosecant of an angle and the cosecant of its supplement are of the same sign, and are *positive.*

Tan. $C A B = \frac{C B}{A B} =$ a positive quantity, as the lines $C B$ and $A B$ are both positive. But tan. $D A B$ (the supplement of $C A B$) $= \frac{D E}{A E} =$ a negative quantity, as $A E$ is negative (Arts. 41 and 42).

Cotan. $C A B = \frac{A B}{B C} =$ a positive quantity;

Cotan. $D A B = \frac{A E}{D E} =$ a negative quantity.

$$\text{Cos. } CAB = \frac{AB}{AC} = \text{a positive quantity;}$$

$$\text{Cos. } DAB = \frac{AE}{AD} = \text{a negative quantity.}$$

$$\text{Sec. } CAB = \frac{AC}{AB} = \text{a positive quantity;}$$

$$\text{Sec. } DAB = \frac{AD}{AE} = \text{a negative quantity.}$$

47. It follows from the preceding article that trigonometrical tables for acute angles can be used for obtuse angles, as the numerical values of the ratios of the one are the same as those of the corresponding ratios of the other.

To find, therefore, the trigonometric ratio of any obtuse angle, we subtract the given angle from 180° and find the corresponding ratio of the remainder.

Where it is important—as in certain computations—to distinguish the obtuse angle from its supplementary acute angle, we retain the sign of the ratio.

EXAMPLE 1. Find from the tables the trigonometric ratios of an angle of 100°.

2. Required from the tables the trigonometric ratios of an angle of 120° 13′ 45″.

3. Required the trigonometric ratios of an angle of 135°.

4. Required the trigonometric ratios of an angle of 150°.

5. Required the trigonometric ratios of an angle of 120°.

6. If the cosine of an angle is − .813467, what is the angle?

Ans. 144° 26′ 10″.

Neglecting the sign, required the obtuse angle whose

7. Cotan. is 2.31821. *Ans.* 156° 39′ 58.4″.

8. Sine is .634167. *Ans.* 140° 38′ 30.3″.

9. Tangent is 1.81511. *Ans.* 118° 51′ 7.4″.

CHAPTER VI.

OBLIQUE-ANGLED TRIANGLES.

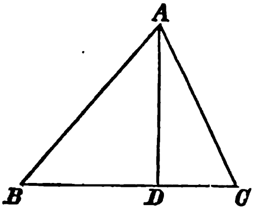
Def. An *oblique-angled triangle* is one which does not contain a right angle. It is, therefore, either an acute-angled or an obtuse-angled triangle.

Art. 48. *In any triangle, the sine of any angle is to the sine of a second angle as the side opposite the first angle is to the side opposite the second angle.*

This principle is sometimes stated in another form, thus: "The sines of the angles of a triangle are proportional to the opposite sides."

In the given triangle, $A\,B\,C$, it is required to prove

$$\frac{\text{Sin. } B}{\text{Sin. } C}=\frac{A\,C}{A\,B}.$$



Suppose that $A\ B\ C$ be a triangle, in which a perpendicular from one of the angles upon the opposite side falls within the triangle, as the perpendicular, $A\,D$, from A upon $B\,C$.

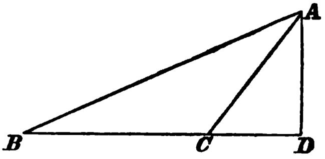
$$\text{Sin. } B=\frac{A\,D}{A\,B} \text{ ((1) Art. 4); also, sin. } C=\frac{A\,D}{A\,C};$$

$$\text{therefore } \frac{\text{sin. } B}{\text{sin. } C}=\frac{\frac{A\,D}{A\,B}}{\frac{A\,D}{A\,C}}=\frac{A\,C}{A\,B}.$$

By drawing a perpendicular from C upon $A\,B$, it can also be proved, in the same way, that

$$\frac{\text{Sin. } A}{\text{Sin. } B} = \frac{B\,C}{A\,C}.$$

Next, let the triangle $A\,B\,C$ be a triangle in which the perpendicular falls without the triangle, on the side produced.



In the figure, $A\,D$, the perpendicular, falls without the triangle. The angle, $A\,C\,B$, is therefore an obtuse angle.

$$\text{Now, sin. } A\,B\,C = \frac{A\,D}{A\,B};$$

$$\text{also, sin. } B\,C\,A = \frac{A\,D}{A\,C} \text{ ((1) Art. 44);}$$

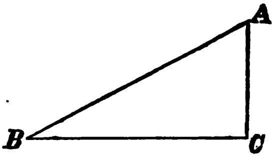
$$\text{therefore } \frac{\text{sin. } A\,B\,C}{\text{sin. } A\,C\,B} = \frac{\frac{A\,D}{A\,B}}{\frac{A\,D}{A\,C}} = \frac{A\,C}{A\,B}.$$

By drawing a perpendicular from B, upon $A\,C$ produced, it may be proved in the same way that

$$\frac{\text{Sin. } B\,A\,C}{\text{Sin. } B\,C\,A} = \frac{B\,C}{A\,B}.$$

Lastly, let $A\,B\,C$ be a right-angled triangle; then also

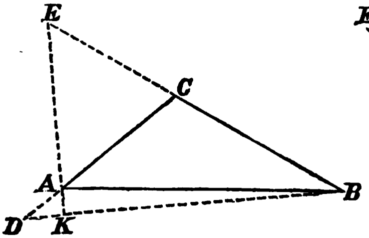
$$\frac{\text{Sin. } B}{\text{Sin. } C} = \frac{A\,C}{A\,B}.$$

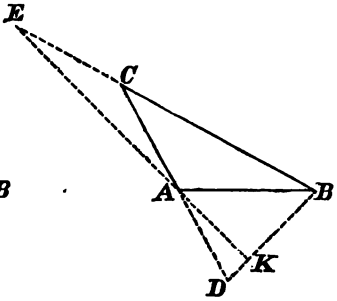


For, since C is a right angle, its sine is 1 (Art. 35). Also

$$\text{sin. } B = \frac{A\,C}{A\,B}; \text{ i. e., } \frac{\text{sin. } B}{1} = \frac{A\,C}{A\,B}, \text{ or } \frac{\text{sin. } B}{\text{sin. } C} = \frac{A\,C}{A\,B}.$$

49. *In any triangle, the* SUM *of any* TWO SIDES *is to their* DIFFERENCE *as the* TANGENT *of* HALF the sum of the OPPOSITE ANGLES is *to the* TANGENT *of* HALF *their* DIFFERENCE.





Let $A\,C\,B$ be any triangle. Then

$$\frac{B\,C+C\,A}{B\,C-C\,A}=\frac{\tan.\ \frac{1}{2}\,(A+B)}{\tan.\ \frac{1}{2}\,(A-B)}.$$

Produce $C\,A$ to D, making $C\,D$ equal to $C\,B$. Produce $B\,C$ to E, making $C\,E$ equal to $C\,A$. Join D and B, by the straight line $D\,B$. Join E and A, by the straight line $E\,A$, and produce $E\,A$ to meet $D\,B$ at K.

By the construction, $B\,E=B\,C+C\,A$
and $A\,D=B\,C-C\,A$.

Now the sum of the angles $C\,A\,B$ and $C\,B\,A$ equals the sum of the angles D and $C\,B\,D$, because each sum is the supplement of the angle $A\,C\,B$ (Euc. 32, I. Ch. 18, I.).

But $C\,D\,B+C\,B\,D=2\,D$ (Euc. 5, I. Ch. 25, I.).

Therefore $D=\frac{1}{2}\,(C\,A\,B+C\,B\,A)=\frac{1}{2}\,(A+B)$.

Also $\frac{1}{2}\,(A+B)+\frac{1}{2}\,(A-B)=A$;

That is, $D + \frac{1}{2}(A - B) = CAB = D + ABD$ (Euc. 32, I. Ch. 18, I. Cor. 1);

therefore, $ABD = \frac{1}{2}(A - B)$.

Again, the angle $CEA = CAE = DAK$;

also $CBK = ADK$;

therefore, $BEK + EBK = DAK + ADK$;

consequently $EKB = EKD$, and each is a right angle, according to the definition of a right angle.

Also the triangles KBE and ADK are similar (Euc. 4, I. Ch. 4, III.).

(1) Now $\frac{AK}{DK} = \text{tan. } D = \text{tan. } \frac{1}{2}(A + B)$;

(2) also $\frac{AK}{KB} = \text{tan. } ABK = \text{tan. } \frac{1}{2}(A - B$;

therefore, dividing (1) by (2), $\frac{KB}{DK} = \frac{\text{tan. } \frac{1}{2}(A+B)}{\text{tan. } \frac{1}{2}(A-B)}$.

But, from the similarity of the triangles, KBE and ADK,

$$\frac{KB}{KD} = \frac{BE}{AD} = \frac{BC + CA}{BC - CA};$$

therefore $\frac{BC + CA}{BC - CA} = \frac{\text{tan. } \frac{1}{2}(A + B)}{\text{tan. } \frac{1}{2}(A - B)}$.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

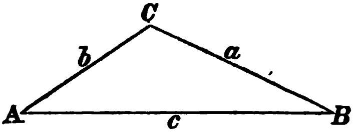
50. To solve an oblique-angled triangle, it is necessary to know *three* parts, of which *one* at least must be a side.

The *three angles* of a triangle determine only its *form*. The same three angles may belong to a number of triangles, of different sides and of different areas, but all similar.

We shall then have different cases of the solution of oblique-angled triangles, which may be classified as follows:

1. When *two angles* and a *side* are given;
2. When *two sides* and an *angle* are given;
3. When the *three sides* are given.

51. *Two angles and a side of an oblique-angled triangle being known, to solve the triangle.*



The third angle is found by subtracting the sum of the given angles from 180°.

The sides are found by the theorem of Art. 48. Thus in the triangle ACB, suppose the side a is given and any two angles as B and A.

Then $C = 180° - (A + B)$.

Also, by Art. 48, $\dfrac{\sin. A}{\sin. B} = \dfrac{a}{b}$; $\therefore b = \dfrac{a \sin. B}{\sin. A}$.

Again, $\dfrac{\sin. C}{\sin. A} = \dfrac{c}{a}$; $\therefore c = \dfrac{a \sin. C}{\sin. A}$.

Solve the triangle when there are given:

Example 1. A side = 121, and the adjacent angles = 15°, and 55° 31′. *Ans.* Sides = 33.219 and 105.797.

2. A side = 14.3, an adjacent angle = 43° 10′ 15″, an opposite angle = 73° 13′ 4″. *Ans.* Sides = 10.219 and 13.3798.

3. A side = 31.57, angle opposite = 55°, and the other sides equal.

4. $a = 15.189$, $B = 75° 10'$, $C = 33° 14' 7''$.

5. $c = 568.9$, $B = 3A$, and $C = 6A$.

6. $b = 7.93$, $A = 3B$, and $B = 2C$.

52. *Two sides and an angle of an oblique-angled triangle being known, to solve the triangle.*

Under this head we have two cases: 1. When the

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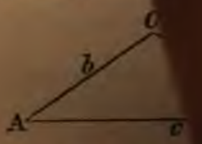

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Then by the theorem of Art. 48,

$$\frac{\sin.\ A}{\sin.\ C}=\frac{a}{c};\ \text{or}\ c=\frac{a\sin.\ C}{\sin.\ A}.$$

Solve the triangle when there are given:

EXAMPLE 1. $c = 13$, $b = 7$, and angle $A = 52°$.

Ans. $a = 10.293$, $C = 95° \ 35' \ 43''$, $B = 32° \ 24' \ 17''$.

2. $c = 12$, $b = 8$, and angle $A = 42°$.

Ans. $a = 8.0818$, $B = 41° \ 28' \ 47''$, $C = 96° \ 31' \ 13''$.

3. $b = 4$, $c = 16$. $A = 55°$.

Ans. $B = 13° \ 26' \ 43''$, $C = 111° \ 33' \ 17''$, $a = 14.092$.

4. $b = 9$, $c = 19$, $A = 50°$.

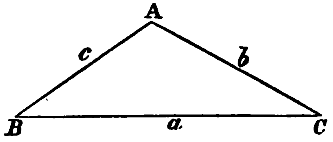
Ans. $a = 14.905$, $B = 27° \ 33' \ 6''$, $C = 102° \ 26' \ 54''$.

5. $a = 11$, $c = 17$, $B = 70°$.

Ans. $b = 16.795$, $A = 37° \ 59' \ 3''$, $C = 72° \ 0' \ 57''$.

6. $a = 10$, $b = 18$, $C = 30°$.
7. $b = 543$, $c = 721$, $A = 65°$.
8. $b = 543$, $c = 721$, $A = 75°$.

54. *Two sides, and an angle opposite one of the given sides, of an oblique-angled triangle being known, to solve the triangle.*



The triangle may be solved by the theorem of Art. 48.

Thus in the triangle $A\,B\,C$, suppose we have given the two sides a and c, and the angle A; then, by Art. 48,

$$\frac{c}{a}=\frac{\sin.\ C}{\sin.\ A};\ \text{therefore}\ \sin.\ C=\frac{c}{a}\sin A.$$

Having C and A, we then find $B = 180° - (A + C)$.

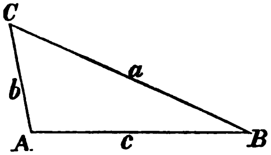
Then (Art. 48), $\frac{b}{a}=\frac{\sin. B}{\sin. A}$; or $b=\frac{a\sin.\ B}{\sin.\ A}$.

55. Of the parts mentioned in Art. 54, *when the side opposite the given angle is less than the other given side, and greater than the product of that side into the sine of the given angle, two triangles can be constructed from the given parts, and therefore two solutions will be possible, both of which will be correct.*

In other cases there will be but *one* solution.

The first case is sometimes spoken of as the *ambiguous* case.

We will consider the cases which are not ambiguous, and then the ambiguous case.



1. Suppose, in the triangle $A\,B\,C$, there are given the two sides, c and a, and the angle A, and that c is *less* than a.

By Art. 48,

$$(1)\ \frac{\sin. C}{\sin. A} = \frac{c}{a};\ \text{therefore}\ \sin. C = \frac{c}{a}\sin. A.$$

But if c is less than a, $\frac{c}{a}$ is less than 1; and, therefore, $\frac{c}{a}\sin. A$ is less than $\sin. A$, or, since $\sin. C$ is equal to $\frac{c}{a}\sin. A$, $\sin. C$ is less than $\sin. A$.

Therefore C is less than A (Art. 8), and must be an *acute* angle. For if C were an obtuse angle, as its sine is less than sin. A, C would be greater than the supplement of A (Art. 45). Consequently C and A, two angles of a triangle, would be together greater than two right angles, which is impossible (Euc. 17, I. Ch. 18, I.). Therefore, in this case, there is no ambiguity.

Or, by geometry, C is less than A (Euc. 18, I. Ch. 26, I.).

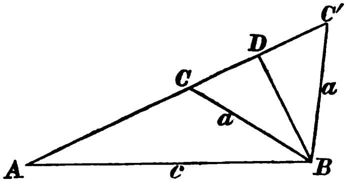
2. Again, suppose $c = a$. Then, as $\frac{c}{a} = 1$, from (1) sin. $C =$ sin. A and $C = A$. C could not be the supplement of A, for then the two angles of a triangle would be equal to two right angles, which is impossible.

By geometry $C = A$ (Euc. 5, I. Ch. 25, I.).

3. Lastly, suppose $a = c$ sin. A.

Substituting this value in (1) sin. $C = 1$; therefore, C is a right angle (Art. 35). And in this case there is no ambiguity.

4. Now, suppose a is less than c, but greater than c sin. A.



If c and a and the angle A are given, and a is greater than c sin. A, then a is greater than $B\,D$ (*see* figure), the perpendicular from B upon $A\,D$, for this perpendicular is equal to c sin. A (Art. 30).

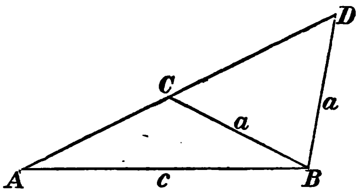
From the equation sin. $C = \frac{c}{a}$ sin. A, as $\frac{c}{a}$ is greater than 1, $\frac{c}{a}$ sin. A, or its equal, sin. C, is greater than sin. A.

Now, A cannot be an obtuse angle when a is less than c, for then would C be also an obtuse angle (Euc. 18, I. Ch. 26, I.), that is, two angles of a triangle would be together greater than two right angles, which is impossible. Therefore, A is an acute angle. Also, as sin. C has been shown to be greater than sin. A, C is an angle greater than A, but may be either an acute angle, or an obtuse angle the supplement of the acute angle, as the sine of an angle and the sine of its supplement are the same in value and in sign (Art. 46).

Thus, in the figure, if a is less than c and greater than c sin. A, we shall have two triangles, $A\,C'\,B$ and $A\,C\,B$, having the angle A and the side c in common, and the side $B\,C'$ equal to $B\,C$, but the angle $B\,C'\,A$ the supplement of $B\,C\,A$.

5. When a is less than c sin. A, there is no triangle formed.

56. *To solve the triangle*, when the parts are given as in the preceding article; that is, *when the side opposite the given angle is less than the other given side, and greater than the product of the sine of the given angle by that side.*



Suppose we have given the sides c and a and the angle A, and that a is less than c, but greater than c sin. A.

From A draw the line $A\,D$, making the given angle with $A\,B\,(c)$, and produce it indefinitely. From B as a centre with a radius equal to a, describe a circle cutting $A\,D$ at C and at D. Join B and C, and B and D, by the straight lines $B\,C$ and $B\,D$ respectively. Then we have two triangles, each of which contains the given parts.

Since $B\,C$ equals $B\,D$, the angle $B\,C\,D$ equals $B\,D\,C$; therefore $A\,C\,B$, which is the supplement of $B\,C\,D$, is also the supplement of $B\,D\,C$.

(1) $\frac{\text{Sin. } D}{\text{Sin. } A} = \frac{c}{a}$; (Art. 48) $\therefore$ $\text{sin. } D = \frac{c}{a} \text{ sin. } A$.

We thus find D. $A\,B\,D = 180° - (A + D)$.

(2) Then $\frac{A\,D}{c} = \frac{\text{sin. } A\,B\,D}{\text{sin. } D}$;

$$\therefore\ A\,D = \frac{c \text{ sin. } A\,B\,D}{\text{sin. D}}.$$

Thus the triangle $A\,B\,D$ is solved.

To solve the triangle $A\,B\,C$, we shall have, as in (1) for sin. D, $\text{sin. } C = \frac{c}{a} \text{ sin. } A$; but as $A\,C\,B$ is the supplement of $A\,D\,B$, after finding D, we subtract D from 180° to find $A\,C\,B$. Then $A\,B\,C = 180° - (A\,C\,B + A)$, or $= D\,C\,B - A$.

Lastly, $\frac{A\,C}{B\,C} = \frac{\text{sin. } A\,B\,C}{\text{sin. } A}$;

$$\therefore\ A\,C = \frac{a \text{ sin. } A\,B\,C}{\text{sin. } A}.$$

Solve an oblique-angled triangle when there are given:

Example 1. $A = 74° \, 45'$, $a = 475$, $b = 432$.

Ans. $B = 61° \, 20' \, 10''$, $C = 43° \, 54' \, 50''$, $c = 341.47$.

2. $B = 67° \, 30'$, $b = 310$, $c = 292$.

3. $C = 41° \, 15' \, 5''$, $c = 891$, $a = 311$.

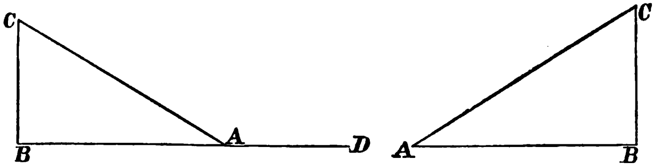
4. $B = 49° \, 30'$, $b = 5$, $c = 6$.

Ans. Two solutions, $A = 64° \, 38' \, 55''$, $C = 65° \, 51' \, 5''$, $a = 5.9422$.
or $16° \, 21' \, 5''$, or $114° \, 8' \, 55''$, or 1.8512.

5. $B = 35^\circ$, $b = 111$, $c = 123$.
Ans. Two solutions, $A = 105^\circ\, 32'\, 12''$, $C = 39^\circ\, 27'\, 48''$, $a = 186.45$.
or $4^\circ\, 27'\, 48''$, or $140^\circ\, 32'\, 12''$, or 15.06.
6. $C = 25^\circ$, $c = 115$, $b = 191$.
Ans. Two solutions, $A = 110^\circ\, 25'\, 10''$, $B = 44^\circ\, 34'\, 50''$, $a = 255.015$.
or $19^\circ\, 34'\, 50''$, or $135^\circ\, 25'\, 10''$, or 91.194.
7. $C = 31^\circ\, 30'$, $c = 115$, $b = 191$.
8. $A = 40^\circ$, $a = 129$, $c = 165$.

57. In order to solve an oblique-angled triangle when its three sides are given, we establish three principles: 1. That *the tangent of an angle is equal to the sine of the angle divided by its cosine;* 2. That *the sine of half an angle is equal to the square root of half the difference between 1 and the cosine of the angle;* and, 3. That *the cosine of half an angle is equal to the square root of half the sum of 1 and the cosine of the angle; sum and difference* being used in the *algebraic* sense.

58. *The tangent of an angle is equal to its sine divided by its cosine.*



Suppose A to be a given angle, acute in the right-hand figure, obtuse in the left-hand figure.

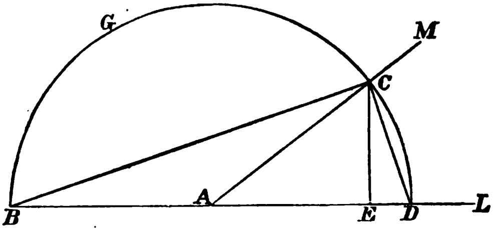
$$\text{Then tan. } A = \frac{\text{sin. } A}{\text{cos. } A}.$$

Make the triangle $A\,B\,C$, right-angled at B, the triangle of reference.

$$\text{Tan. } A = \frac{CB}{AB} \text{ ((2) Art. 4 and (2) Art. 44)} = \frac{\frac{CB}{AC}}{\frac{AB}{AC}}$$

$$= \frac{\text{sin. } A}{\text{cos. } A} \text{ ((1) and (4) Articles 4 and 44).}$$

59. *The sine of half an angle is equal to the square root of half the difference between 1 and the cosine of the whole angle.*



1. Let MAL be an acute angle; then

$$\text{sin. } \tfrac{1}{2} A = \sqrt{\frac{1 - \text{cos. } A}{2}}.$$

With A as centre and any radius AC, describe a semicircle, BGD, meeting AL in D, and AL produced in B, and cutting AM in C. Draw the straight lines BC and CD, and from C draw CE perpendicular to AD, meeting AD at E.

The angle $B = \frac{1}{2} A$ (Euc. 5 and 32, I. Ch. 25, and Cor. I. 18, I.);

$$\text{sin. } \tfrac{1}{2} A = \text{sin. } B = \frac{CE}{CB};$$

$$(\text{sin. } B)^2 \text{ (written sin. }^2 B) = \frac{CE^2}{CB^2} = \frac{BE \times ED}{BE \times BD}$$

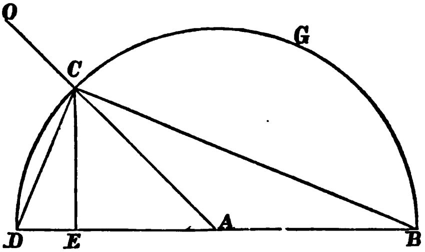
$$\text{(Euc. 8, VI. Ch. 13, III.)} = \frac{ED}{BD};$$

therefore $\sin.^2 B = \frac{ED}{BD} = \frac{AD - AE}{2\,AD} = \frac{1}{2}\left(1 - \frac{AE}{AC}\right)$
$= \frac{1}{2}(1 - \cos. A)$;

therefore $\sin. \frac{1}{2} A = \sin. B = \sqrt{\frac{1 - \cos. A}{2}}$.

2. Let us take the obtuse angle BAO; then, also,

$$\sin. \tfrac{1}{2} A = \sqrt{\frac{1 - \cos. A}{2}}.$$



From A as centre, with radius AC, describe the semicircle BGD, and complete the figure, as in the previous article.

The angle $BDC = \frac{1}{2} BAC$;

therefore $\sin. \frac{1}{2} A = \sin. BDC = \frac{CE}{CD}$.

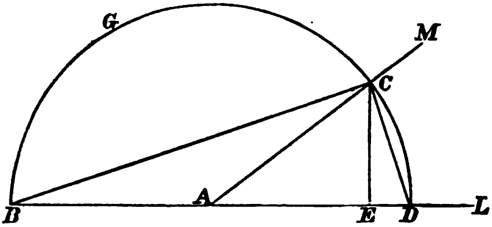
$$\text{Sin.}^2 BDC = \frac{CE^2}{CD^2} = \frac{DE \times BE}{DE \times BD} = \frac{BE}{BD}$$
$$= \frac{BA + AE}{2\,BA} = \tfrac{1}{2}\left(1 + \frac{AE}{AC}\right).$$

Now AE is a minus quantity (Art. 43), and $\frac{AE}{AC} = -\frac{-AE}{AC} = -\cos. A$ ((4) Art. 44).

Therefore $\sin. \tfrac{1}{2} A = \sin. B D C = \sqrt{\dfrac{1 - \cos. A}{2}}$.

60. *The cosine of half an angle is equal to the square root of one-half the sum of 1 and the cosine of the whole angle.*

First, let the angle be an acute angle, as the angle $M A L$.



Then $\cos. \tfrac{1}{2} A = \sqrt{\dfrac{1 + \cos. A}{2}}$.

Construct the figure as for the first part of Art. 59.

$$\text{Cos. } \tfrac{1}{2} A = \cos. B = \frac{B E}{B C}.$$

$$\text{Cos.}^2 B = \frac{B E^2}{B C^2} = \frac{B E^2}{B E \times B D} = \frac{B E}{B D} = \frac{B A + A E}{2 A C}$$

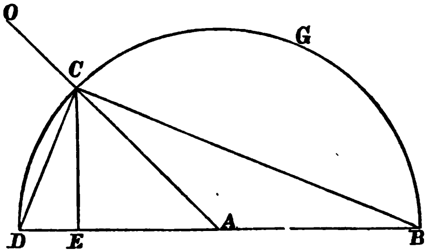
$$= \tfrac{1}{2}\left(1 + \frac{A E}{A C}\right).$$

Now $\dfrac{A E}{A C} = \cos. A$ ((4) Art. 4);

therefore $\cos. \tfrac{1}{2} A = \cos. B = \sqrt{\dfrac{1 + \cos. A}{2}}$.

Next let A be an obtuse angle, as the angle $B A O$.

Then also $\cos. \tfrac{1}{2} A = \sqrt{\dfrac{1 + \cos. A}{2}}$.



Construct the figure as for the second part of Art. 59.

$$\text{Cos. } \tfrac{1}{2} A = \text{cos. } BDC = \frac{DE}{CD}.$$

$$\text{Cos. }^2 BDC = \frac{DE^2}{CD^2} = \frac{DE^2}{DE \times BD} = \frac{DE}{BD}$$

$$= \frac{AD - AE}{2\,AD} = \tfrac{1}{2}\left(1 - \frac{AE}{AC}\right).$$

But $\text{cos. } A = \frac{AE}{AC}$, and is itself a minus quantity (Art. 43, (4) Art. 45), and therefore $-\frac{AE}{AC}$ is $+ \text{cos. } A$.

$$\text{Therefore cos. } \tfrac{1}{2} A = \text{cos. } BDC = \sqrt{\frac{1 + \text{cos. } A}{2}}.$$

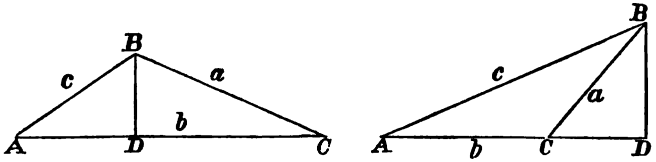
61. *The three sides of a triangle being known, to solve the triangle.*

Let ABC be an oblique-angled triangle, of which the sides a, b, and c are given. It is required to find the angles.

From B draw a perpendicular BD to AC, or AC produced.

(1) $AD = AB$ cos. $A = c$ cos. A (Art. 30).

(2) Also, $a^2 + 2\,b \times AD = b^2 + c^2$ (Euc. 13, II. Ch. 15, III.).



In (2) substitute the value of $A\,D$ already found in (1) and $a^2 + 2\,b\,c\ \cos. A = b^2 + c^2$.

(3) Therefore $\cos. A = \dfrac{b^2 + c^2 - a^2}{2\,b\,c}.$

(4) Subtracting each member of (3) from 1, and we have $1 - \cos. A = \dfrac{2\,b\,c - b^2 - c^2 + a^2}{2\,b\,c} = \dfrac{a^2 - (b - c)^2}{2\,b\,c}$

$$= \frac{(a + c - b)\ (a + b - c)}{2\,b\,c}.$$

Now let $\dfrac{a + b + c}{2} = s;$

then $\dfrac{a + c - b}{2} = s - b;\ \dfrac{a + b - c}{2} = s - c;$ and $\dfrac{b + c - a}{2} = s - a.$

Dividing (4) by 2, extracting the square root of both members, and substituting the values given for $\dfrac{a + c - b}{2}$, and $\dfrac{a + b - c}{2}$,

$$\sqrt{\frac{1 - \cos. A}{2}} = \sqrt{\frac{(s - b)\ (s - c)}{b\,c}}.$$

But $\sqrt{\dfrac{1 - \cos. A}{2}} = \sin. \tfrac{1}{2} A$ (Art. 59).

(5) Therefore $\sin.\ \frac{1}{2} A = \sqrt{\frac{(s-b)\ (s-c)}{b\ c}}.$

Again, adding 1 to both members of (3),

$$(6)\quad 1 + \cos. A = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2\ b\ c}$$
$$= \frac{(a+b+c)\ (b+c-a)}{2bc}.$$

Dividing (6) by 2, extracting the square root, and substituting values for $\frac{a+b+c}{2}$ and $\frac{b+c-a}{2}$,

$$\sqrt{\frac{1 + \cos. A}{2}} = \sqrt{\frac{s\ (s-a)}{b\ c}}.$$

But $\sqrt{\frac{1+\cos. A}{2}} = \cos.\ \frac{1}{2} A$ (Art. 60).

(7) Therefore $\cos. \frac{1}{2} A = \sqrt{\frac{s\,(s-a)}{b\ c}}.$

Dividing (5) by (7) we have

$$(8)\quad \frac{\text{Sin.}\ \frac{1}{2} A}{\text{Cos.} \frac{1}{2} A} = \tan.\ \frac{1}{2} A = \sqrt{\frac{(s-b) \cdot (s-c)}{s\ (s-a)}}.$$

We can prove in a similar manner:

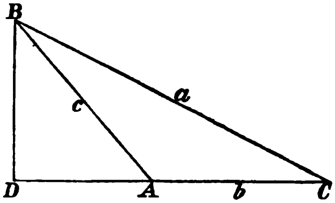
$\sin.\ \frac{1}{2} B = \sqrt{\frac{(s-c)\ (s-a)}{c\,a}}$; $\cos. \frac{1}{2} B = \sqrt{\frac{s\ (s-b)}{c\,a}}$;

and $\tan.\ \frac{1}{2} B = \sqrt{\frac{(s-c)\ (s-a)}{s\ (s-b)}}.$

Also:

$\sin.\ \frac{1}{2}\ C = \sqrt{\frac{(s-a)\ (s-b)}{a\ b}}$; $\cos. \frac{1}{2}\ C = \sqrt{\frac{s\ (s-c)}{a\ b}}$;

and $\tan.\ \frac{1}{2}\ C = \sqrt{\frac{(s-a)\ (s-b)}{s\,(s-c)}}.$



62. If the angle, as A, in the figure, be an obtuse angle, the expressions in (5), (7), and (8), of the preceding article will still be of the same form.

For we shall have:

(1) $a^2 - 2b \times AD = b^2 + c^2$ (Euc. 12, II. Ch. 16, III.).

Now $AD = c \times \cos. BAD = c \times - \cos. A$ (Art. 46), and substituting this value in (1) we shall have $\cos. A = \dfrac{b^2 + c^2 - a^2}{2bc}$, the same expression as in equation (3) of the preceding article. Therefore we shall have the same expressions for sin. $\frac{1}{2} A$, cos. $\frac{1}{2} A$, tan. $\frac{1}{2} A$, as in the preceding article.

Find the angles of a triangle when—

Example 1. $a = 6$, $b = 5$, $c = 4$.
Ans. $A = 82° 49' 10''$, $C = 41° 24' 34''$. $B = 55° 46' 16''$.

2. $a = 11$, $b = 13$, $c = 16$. *Ans.* $A = 43° 2' 56''$, $C = 83° 10' 22''$. $B = 53° 46' 42''$.

3. $a = 25$, $b = 26$, $c = 27$. *Ans.* $A = 56° 15' 4''$, $C = 63° 53' 46''$. $B = 59° 51' 10''$.

4. $a = 222$, $b = 318$, $c = 406$.
Ans. $A = 32° 57' 7''$, $C = 95° 51' 55''$. $B = 51° 10' 58''$.

5. $a = 400$, $b = 340$, $c = 250$.

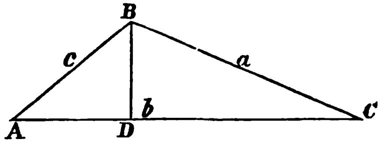
6. $a = 1410$, $b = 2160$, $c = 3142$.

7. $a = 520$, $b = 348$, $c = 192$.

8. $a = 412$, $b = 562$, $c = 330$.

63. *Second method of solving a triangle when the three sides are known.*

Let ABC be any triangle of which the three sides



are known. From B draw a perpendicular BD upon AC, the longest side, dividing the triangle into two right-angled triangles, ABD and BDC.

(1) $BC^2 = CD^2 + BD^2$ (Euc. 47, I. Ch. 14; III.).

(2) $AB^2 = AD^2 + BD^2$.

Subtracting equation (2) from equation (1),

$$BC^2 - AB^2 = CD^2 - AD^2.$$

Factoring,

$$(BC + AB)(BC - AB) = AC(CD - DA).$$

$$(3)\quad CD - DA = \frac{(BC + AB)(BC - AB)}{AC}$$

$$= \frac{(a + c)(a - c)}{b}.$$

Equation (3) will give the difference of the segments of the base. *Half the base* (or half the sum of the segments) *added to half the difference of the segments of the base* will give the *greater* segment, CD; and *half the difference of the segments* of the base *subtracted from half the base* will give the *smaller* segment, AD. Then, in each of the two right-angled triangles ABD and BCD, we have the hypotenuse and a side given, to solve the triangle (Art. 26).

Solve the triangle when the sides are given:

EXAMPLE 1. $a = 8$, $b = 6$, $c = 4$.

Ans. Segments of $a = 5.25$, $A = 104° \, 28' \, 39''$, $B = 46° \, 34' \, 3''$,
2.75, $C = 28° \, 57' \, 18''$.

2. $a = 219$, $b = 91$, $c = 245$.

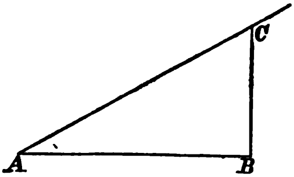
Ans. Segments of $c = 203.479$, $A = 62° \, 51' \, 11''$, $C = 95° \, 26' \, 45''$.
41.521, $B = 21° \, 42' \, 4''$.

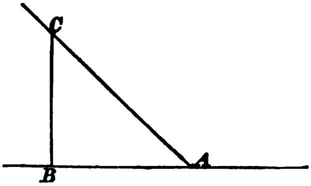
3. $a = 1140$, $b = 718.9$, $c = 627$.

CHAPTER VII.

RELATIONS OF THE TRIGONOMETRICAL RATIOS TO EACH OTHER.—TRIGONOMETRICAL RATIOS OF TWO ANGLES.

ART. 64. *The trigonometrical ratios of an angle interchangeable.*





Let $B\,A\,C$ be the triangle of reference for the angle A.

$$\text{Sin. } A = \frac{B\,C}{A\,C};\ \text{cos. } A = \frac{A\,B}{A\,C}.$$

Squaring and adding,

$$(a)\ \text{Sin.}^2 A + \text{cos.}^2 A = \frac{B\,C^2 + A\,B^2}{A\,C^2} = 1;$$

$$\text{sin. } A = \pm\sqrt{1 - \text{cos.}^2 A};$$

$$\text{cos. } A = \pm\sqrt{1 - \text{sin.}^2 A}.$$

$$(b)\ \text{Tan. } A = \frac{\text{sin. } A}{\text{cos. } A}\ \text{(Art. 58).}$$

$$(c)\ \text{Sec. } A = \frac{A\,C}{A\,B} = \frac{1}{\frac{A\,B}{A\,C}} = \frac{1}{\text{cos. } A}.$$

(*d*) $$\text{Cot. } A = \frac{AB}{BC} = \frac{\frac{AB}{AC}}{\frac{BC}{AC}} = \frac{\cos. A}{\sin. A}.$$

(*e*) $$\text{Cosec. } A = \frac{AC}{BC} = \frac{1}{\frac{BC}{AC}} = \frac{1}{\sin. A}.$$

(*f*) $$\text{Tan. } A = \frac{BC}{AB} = \frac{1}{\frac{AB}{BC}} = \frac{1}{\cot. A}.$$

(*f*) also follows from (*b*) and (*d*).

(*g*) $$\text{Sec. } A = \frac{AC}{AB} = \sqrt{\frac{AC^2}{AB^2}} = \sqrt{\frac{AB^2 + BC^2}{AB^2}}$$
$$= \sqrt{1 + \tan.^2 A}.$$

(*h*) $$\text{Cosec. } A = \frac{AC}{BC} = \sqrt{\frac{AC^2}{BC^2}} = \sqrt{\frac{BC^2 + AB^2}{BC^2}}$$
$$= \sqrt{1 + \cot.^2 A}.$$

65. By means of the equations established in the preceding article, it is possible to express all the trigonometrical ratios in terms of any one ratio.

Thus to express the trigonometrical ratios in terms of the tangent:

$$\text{Sin. } A = \frac{1}{\text{cosec. } A} \;((e)\text{ Art. } 64) = \frac{1}{\sqrt{1 + \cot.^2 A}}$$
$$= \frac{1}{\sqrt{1 + \frac{1}{\tan.^2 A}}} = \frac{\tan. A}{\sqrt{1 + \tan.^2 A}}.$$

$$\text{Cos. } A = \frac{1}{\text{sec. } A} = \frac{1}{\sqrt{1 + \text{tan.}^2 A}}.$$

$$\text{Sec. } A = \sqrt{1 + \text{tan.}^2 A}.$$

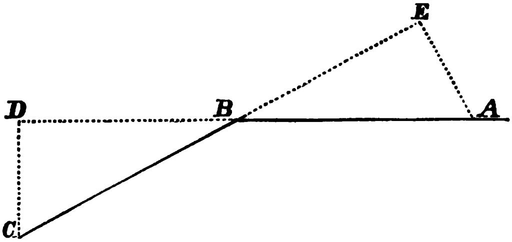
$$\text{Cosec. } A = \sqrt{1 + \text{cot.}^2 A} = \sqrt{1 + \frac{1}{\text{tan.}^2 A}}$$

$$= \frac{\sqrt{1 + \text{tan.}^2 A}}{\text{tan. } A}.$$

$$\text{Cot. } A = \frac{1}{\text{tan. } A}.$$

1. Given sin. $A = \frac{1}{3}$; find the remaining trigonometric ratios for A.
2. Given cos. $B = \frac{2}{3}$; find the remaining trigonometric ratios for B.
3. Given tan. $C = \frac{3}{2}$; find the remaining trigonometric ratios for C.
4. Given sec. $D = 3$; find the remaining trigonometric ratios for D.
5. Given cot. $E = \frac{7}{8}$; find the remaining trigonometric ratios for E.
6. Given cosec. $F = \frac{7}{4}$; find the remaining trigonometric ratios for F.
7. Given cos. $G = -\frac{3}{5}$; find the remaining trigonometric ratios for G.
8. Given tan. $H = \frac{10}{-3}$; find the remaining trigonometric ratios for H.

66. The definitions of the trigonometrical ratios, as



applied to an acute angle and to an obtuse angle, may be extended to any angle whatever.

Thus, if $A\,B\,C$ (the salient angle) be an angle *great-*

er than two right angles, we form the triangle of reference, BCD, by drawing a perpendicular, from any point C in one side of the angle, to the other side, AB produced (Art. 3). Then—

(1) Sin. $ABC = \frac{CD}{BC} =$ a negative quantity (Art. 41).

(2) Tan. $ABC = \frac{CD}{BD} =$ a positive quantity.

(3) Sec. $ABC = \frac{CB}{BD} =$ a negative quantity.

(4) Cos. $ABC = \frac{BD}{BC} =$ a negative quantity.

(5) Cot. $ABC = \frac{BD}{CD} =$ a positive quantity.

(6) Cosec. $ABC = \frac{BC}{CD} =$ a negative quantity.

Also, if we draw a perpendicular from A upon CB produced, the triangle ABE will be the triangle of reference, and will be similar to CBD. (Euc. 4, VI. Ch. 4, III. Cor.)

Consequently—

(7) Sin. $ABC = \frac{AE}{AB} = \frac{CD}{BC}$.

(8) Cos. $ABC = \frac{BE}{AB} = \frac{BD}{BC}$, etc.

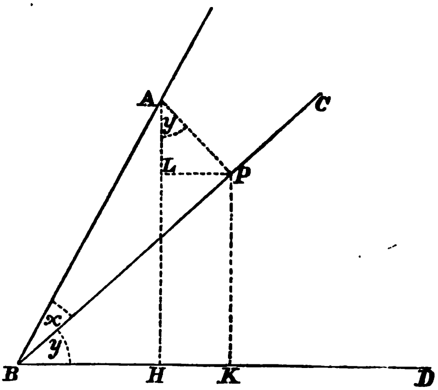
67. *To express the sine and the cosine of the sum of two angles, in terms of the sines and cosines of the given angles.*

Denote one of the angles by x, and the other by y, then we shall prove—

(*a*) Sin $(x+y) =$ sin. x cos. $y+$ sin. y cos. x.

(*b*) Cos. $(x+y) =$ cos. x cos. $y -$ sin. x sin. y.

First, let both angles be *acute*. Let ABC and CBD be two acute angles, whose sum is ABD. It is required to find the sine of ABD in terms of the sines and cosines of ABC, and of CBD.



Denote the angle ABC by x, and CBD by y.

In AB, take any point A, and from A draw AP perpendicular to BC, and AH perpendicular to BD. From P, the foot of the perpendicular from A upon BC, draw PK perpendicular to BD, and PL perpendicular to AH.

The angle LAP equals CBD, that is, y.

$$\text{Sin.}\,ABD = \frac{AH}{AB} = \frac{AL+PK}{AB} = \frac{AL}{AB} + \frac{PK}{AB}$$

$$= \frac{AP}{AB} \times \frac{AL}{AP} + \frac{PK}{BP} \times \frac{BP}{AB}.$$

Now, sin. $ABD =$ sin $(x+y)$;

Also, $\frac{AP}{AB} = \sin. x$; $\frac{AL}{AP} = \cos. y$; $\frac{PK}{BP} = \sin. y$; and $\frac{BP}{AB} = \cos. x$;

Therefore, $\sin. (x + y) = \sin. x \cos. y + \sin y \cos. x$.

Again, $\cos. (x + y) = \cos. x \cos. y - \sin x \sin y$.

For, the same construction being made—

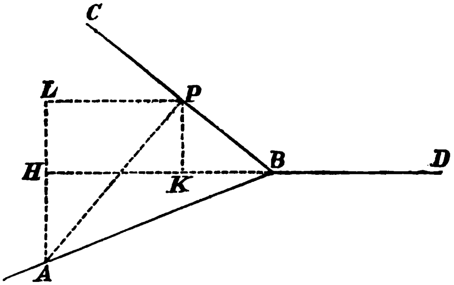
$$\text{Cos. } ABD = \frac{BH}{AB} = \frac{BK - LP}{AB} = \frac{BK}{AB} - \frac{LP}{AB}$$

$$= \frac{BP}{AB} \times \frac{BK}{BP} - \frac{AP}{AB} \times \frac{LP}{AP}.$$

But $\cos. ABD = \cos. (x + y)$;

Also, $\frac{BP}{AB} = \cos. x$; $\frac{BK}{BP} = \cos. y$; $\frac{AP}{AB} = \sin. x$; and $\frac{LP}{AP} = \sin. y$;

Therefore $\cos. (x + y) = \cos. x \cos. y - \sin. x \sin. y$.



Next, let one of the angles, ABC, be an acute angle, and the other, CBD, be an obtuse angle. Denote ABC by x, and CBD by y.

From A (any point in AB), draw AP perpendicular to BC, AH perpendicular to BD produced. From P, where AP meets BC, draw PK perpendicular to BD produced, and draw PL perpendicular to AH produced.

The salient angle, ABD, is the sum of ABC and CBD, and equals $(x+y)$. ABH is the triangle of reference for ABD (Art. 66). Also (Euc. 4, 6. Ch. 7, III.), LAP, which is similar to PKB, (the triangle of reference for CBD, or y), may be used for the triangle of reference for y. The angle LAP is equal to the angle PBK.

By Art. 41, the lines AH, AL, and BK, BH, are negative. We shall put the minus sign before them to indicate the fact. Now,

$$\text{Sin. } ABD = \frac{-AH}{AB} \text{ (Art. 66)} = \frac{-AL+PK}{AB}$$

$$= \frac{-AL}{AB} + \frac{PK}{AB} = \frac{AP}{AB} \times \frac{-AL}{AP} + \frac{PK}{PB} \times \frac{PB}{AB}.$$

But sin. $ABD = \sin.(x+y)$;

$$\text{Also } \frac{AP}{AB} = \sin. x; \ \frac{-AL}{AP} = \frac{-BK}{BP} \text{ ((4) Art. 44)}$$

$$= \cos. y; \ \frac{PK}{PB} = \sin. y; \text{ and } \frac{PB}{AB} = \cos. x.$$

Therefore $\sin.(x+y) = \sin. x \cos. y + \sin. y \cos. x.$

$$\text{Again, cos. } ABD = \frac{-BH}{AB} = \frac{-BK-PL}{AB}$$

$$= \frac{BP}{AB} \times \frac{-BK}{BP} - \frac{AP}{AB} \times \frac{PL}{AP}.$$

But cos. $ABD = \cos.(x+y)$;

Also, $\frac{BP}{AB} = \cos. x$; $\frac{-BK}{BP} = \cos. y$ ((4) Art. 44); $\frac{AP}{AB} = \sin. x$, and $\frac{PL}{AP} = \frac{PK}{PB} = \sin. y$;

Therefore, $\cos. (x + y) = \cos. x \cos. y - \sin. x \sin. y$.

If both x and y are obtuse, it may be proved in a similar manner that—

$\text{Sin.} (x + y) = \sin. x \cos. y + \sin. y \cos. x$.

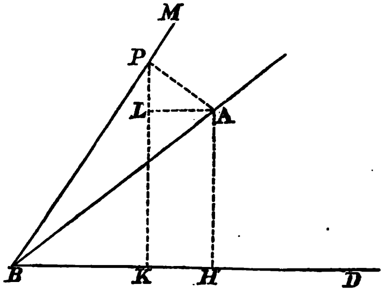
$\text{Cos.} (x + y) = \cos. x \cos. y - \sin. x \sin. y$.

68. *To express the sine and the cosine of the difference of two angles in terms of the sines and cosines of the given angles.*

Denote one of the angles by x, and the other by y; then,

(*a*) $\text{Sin.} (x - y) = \sin. x \cos. y - \sin. y \cos. x$.

(*b*) $\text{Cos.} (x - y) = \cos. x \cos. y + \sin. x \sin. y$.



First, let both angles be acute.

In the figure let the angle MBD be denoted by x, and the angle MBA by y; then ABD will be $(x - y)$.

From A, any point in AB, draw AP perpendicular to BM, and AH perpendicular to BD. From P, the foot of the perpendicular AP, draw PK perpendicular to BD. From A, draw AL parallel to BD. AL will be perpendicular to PK.

The angle LPA is equal to MBD, that is, to x.

$$\text{Sin. } ABD = \frac{AH}{AB} = \frac{PK - PL}{AB} = \frac{PK}{AB} - \frac{PL}{AB}$$
$$= \frac{PK}{PB} \times \frac{PB}{AB} - \frac{AP}{AB} \times \frac{PL}{AP}.$$

Now, sin. $ABD = $ sin. $(x - y)$;

also, $\frac{PK}{PB} = $ sin. x; $\frac{PB}{AB} = $ cos. y; $\frac{AP}{AB} = $ sin. y;

and $\frac{PL}{AP} = $ cos. x;

Therefore, sin. $(x - y) = $ sin. x cos. y $-$ sin. y cos. x.

$$\text{Cos. } ABD = \frac{BH}{AB} = \frac{BK + AL}{AB} = \frac{BK}{AB} + \frac{AL}{AB}$$
$$= \frac{BK}{BP} \times \frac{BP}{BA} + \frac{AL}{AP} \times \frac{AP}{AB}.$$

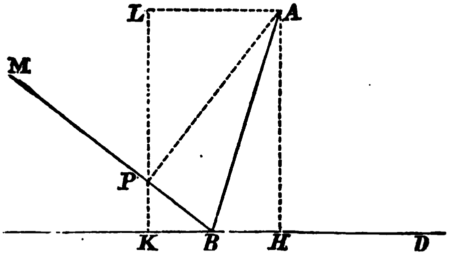
But cos. $ABD = $ cos. $(x - y)$;

also, $\frac{BK}{BP} = $ cos. x; $\frac{BP}{BA} = $ cos. y; $\frac{AL}{AP} = $ sin. x;

and $\frac{AP}{AB} = $ sin. y;

Therefore, cos. $(x - y) = $ cos. x cos. y $+$ sin. x sin. y.

Next, let one angle be obtuse and the other acute.



In the figure, let MBD, the obtuse angle, be denoted by x, and let ABM be the acute angle, denoted by y. Then ABD is $(x-y)$.

The construction is similar to the construction of the preceding figure.

From A, any point in AB, draw AH perpendicular to BD, and AP perpendicular to BM. From P, the foot of the perpendicular AP, draw PK perpendicular to BD produced. Also, from A draw AL parallel to BD, and meeting PK produced through P, at L. The angle LPA is equal to PBK, and the triangle LPA is similar to the triangle PBK, the triangle of reference for MBD, that is, x.

$$\text{Sin. } ABD = \frac{AH}{AB} = \frac{PK+PL}{AB}$$

$$= \frac{PK}{PB} \times \frac{PB}{AB} + \frac{AP}{AB} \times \frac{PL}{AP}.$$

Now, $\sin. ABD = \sin. (x-y)$;

also, $\dfrac{PK}{PB} = \sin. x$; $\dfrac{PB}{AB} = \cos. y$; and $\dfrac{AP}{AB} = \sin. y$.

Again, $-\dfrac{PL}{AP} = \dfrac{-KB}{PB} = \cos. x$ ((4) Art. 44);

therefore, $\dfrac{PL}{AP} = -\cos. x$.

Therefore, $\sin. (x-y) = \sin. x \cos. y - \sin. y \cos. x$.

$$\text{Cos. } ABD = \frac{BH}{AB} = \frac{-KB+AL}{AB}$$

$$= \frac{-KB}{PB} \times \frac{PB}{AB} + \frac{AL}{AP} \times \frac{AP}{AB}.$$

Now, $\cos. ABD = \cos. (x-y)$;

also, $\frac{-KB}{PB} = \cos. x$; $\frac{PB}{AB} = \cos. y$; $\frac{AL}{AP} = \frac{PK}{PB} = \sin. x$; and $\frac{AP}{AB} = \sin. y$.

Therefore, $\cos. (x - y) = \cos. x \cos. y + \sin. x \sin. y$.

If both x and y are obtuse, it may be proved, in a similar manner,

$\sin. (x - y) = \sin. x \cos. y - \sin. y \cos. x$.

$\cos. (x - y) = \cos. x \cos. y + \sin. x \sin. y$.

69. By addition of equation (*a*) of Art. 67 to equation (*a*) of Art. 68, we have

(*a*) $\sin. (x + y) + \sin. (x - y) = 2 \sin. x \cos. y$.

By subtraction, we have

(*b*) $\sin. (x + y) - \sin. (x - y) = 2 \sin. y \cos. x$.

By addition of equation (*b*) of Art. 67 to equation (*b*) of Art. 68, we have

(*c*) $\cos. (x + y) + \cos. (x - y) = 2 \cos. x \cos. y$.

By subtraction, we have

(*d*) $\cos. (x - y) - \cos. (x + y) = 2 \sin. x \sin. y$.

70. In equations of preceding article, let $x + y = a$ and $x - y = b$.

Then, finding values of x and y in terms of a and b,

$$x = \frac{a+b}{2}; \; y = \frac{a-b}{2}.$$

Substituting these values in the preceding equations, in order:

(*a*) $\sin. a + \sin. b = 2 \sin. \frac{a+b}{2} \cos. \frac{a-b}{2}$.

(*b*) $\sin. a - \sin. b = 2 \sin. \frac{a-b}{2} \cos. \frac{a+b}{2}$.

(*c*) $\cos. a + \cos. b = 2 \cos. \frac{a+b}{2} \cos. \frac{a-b}{2}.$

(*d*) $\cos. b - \cos. a = 2 \sin. \frac{a+b}{2} \sin. \frac{a-b}{2}.$

The above formulæ are useful in *logarithmic* calculations.

71. Dividing (*a*) of the preceding article by (*b*), we have

$$(a) \quad \frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \tan. \frac{a+b}{2} \cot. \frac{a-b}{2}$$

$$= \frac{\tan. \frac{a+b}{2}}{\tan. \frac{a-b}{2}}.$$

Dividing (*c*) of the preceding article by (*d*), we have

$$(b) \quad \frac{\cos. a + \cos. b}{\cos. b - \cos. a} = \cot. \frac{a+b}{2} \cot. \frac{a-b}{2}$$

$$= \frac{\cot. \frac{a+b}{2}}{\tan. \frac{a-b}{2}}.$$

72. $\text{Tan.} (x + y) = \frac{\sin. (x + y)}{\cos. (x + y)}$

$$= \frac{\sin. x \cos. y + \sin. y \cos. x}{\cos. x \cos. y - \sin. x \sin. y}.$$

Divide numerator and denominator of the last fraction by $\cos. x \cos. y$:

$$(a) \quad \tan. (x + y) = \frac{\tan. x + \tan. y}{1 - \tan. x \tan. y}.$$

In a similar manner it can be shown—

(*b*) $\tan.(x-y)=\dfrac{\tan.x-\tan.y}{1+\tan.x\tan.y}$.

Again, as $\cot.(x+y)=\dfrac{1}{\tan.(x+y)}$ ((*f*) Art. 64),

(*c*) $\cot.(x+y)=\dfrac{1-\tan.x\tan.y}{\tan.x+\tan.y}$

$$=\frac{1-\dfrac{1}{\cot.x}\times\dfrac{1}{\cot.y}}{\dfrac{1}{\cot.x}+\dfrac{1}{\cot.y}}=\frac{\cot.x\cot.y-1}{\cot.y+\cot.x}.$$

(*d*) $\cot.(x-y)=\dfrac{\cot.x\cot.y+1}{\cot.y-\cot.x}$.

(*c*) may also be proved directly, thus:

$$\cot.(x+y)=\frac{\cos.(x+y)}{\sin.(x+y)}=\frac{\cos.x\cos.y-\sin.x\sin.y}{\sin.x\cos.y+\sin.y\cos.x}.$$

Divide numerator and denominator of last fraction by $\cos.x\cos.y$.

$$\cot.(x+y)=\frac{1-\tan.x\tan.y}{\tan.x+\tan.y}.$$

Divide numerator and denominator of same fraction by $\sin.x\sin.y$.

$$\cot.(x+y)=\frac{\cot.x\cot.y-1}{\cot.y+\cot.x}.$$

(*d*) may be proved in a similar manner.

73. *Sine and cosine of an angle in terms of the sine and cosine of half the angle, and in terms of tangent of half the angle.*

In equations (a) and (b) of Art. 67 let $y = x$, then

$$(a)\ \sin. 2x = 2 \sin. x \cos. x = \frac{\dfrac{2 \sin. x}{\cos. x}}{\dfrac{1}{\cos.^2 x}} = \frac{2 \tan. x}{\sec.^2 x}$$

$$= \frac{2 \tan. x}{1 + \tan.^2 x}.$$

$$(b)\ \cos. 2x = \cos.^2 x - \sin.^2 x = 1 - 2 \sin.^2 x$$

$$= 2 \cos.^2 x - 1$$

$$= \frac{1 - \dfrac{\sin.^2 x}{\cos.^2 x}}{\dfrac{1}{\cos.^2 x}} = \frac{1 - \tan.^2 x}{1 + \tan.^2 x}.$$

74. *Tangent and cotangent of an angle in terms of the tangent and cotangent of half the angle.*

In equations (a) and (c) of Art. 72 let $y = x$, then

$$(a)\ \tan. 2x = \frac{2 \tan. x}{1 - \tan.^2 x} = \frac{\dfrac{2}{\cot. x}}{1 - \dfrac{1}{\cot.^2 x}} = \frac{2 \cot. x}{\cot.^2 x - 1}.$$

$$(b)\ \cot. 2x = \frac{1 - \tan.^2 x}{2 \tan. x} = \frac{\cot.^2 x - 1}{2 \cot. x}.$$

75. *Sine, cosine, and tangent of half an angle in terms of the cosine of the angle.*

From (b) of Art. 73 we have, by transposition, $2 \sin.^2 x = 1 - \cos. 2x$; and $2 \cos.^2 x = 1 + \cos. 2x$.

$$(a)\ \text{Therefore, } \sin. x = \sqrt{\frac{1 - \cos. 2x}{2}};$$

$$(b)\ \cos. x = \sqrt{\frac{1 + \cos. 2x}{2}};$$

equations which have already been derived by geometric methods (Articles 59 and 60).

Dividing (*a*) by (*b*), we have

$$(c)\ \tan. x = \sqrt{\frac{1 - \cos. 2x}{1 + \cos. 2x}}.$$

EXAMPLE 1. Find $\sin. 3x = \sin. (2x + x)$ in terms of $\sin. x$.

2. Find $\cos. 3x$ in terms of $\cos. x$.

3. Find $\cos. 4x$ in terms of $\sin. x$.

4. Find $\cos. 6x$ in terms of $\sin. x$.

5. Find $\sin. 6x$ in terms of $\sin. x$ and $\cos. x$.

6. Prove $\sin. (30° + x) + \sin. (60° + x) = \frac{1}{2}(1 + \sqrt{3})(\sin. x + \cos. x)$.

7. Prove $\cos. (30° + x) + \cos. (60° + x) = \frac{1}{2}(1 + \sqrt{3})(\cos. x - \sin. x)$.

8. Prove $\sin. (45° + x) + \cos. (45° + x) = \sqrt{2} \cos. x$.

9. Prove $\sin. (45° + x) + \cos. (45° + x) = \sin. (45° - x) + \cos. (45° - x)$.

10. Prove $\tan. (45° + x) + \tan. (45° - x) = \frac{2(1 + \tan.^2 x)}{(1 - \tan.^2 x)}$.

11. Prove $\tan. (x + 45°) + \tan. (x - 45°) = \frac{4 \tan. x}{1 - \tan.^2 x}$.

12. Show that $\sin. 75° + \sin. 15° = \sqrt{\frac{3}{2}}$.

13. Show that $\sin. 75° - \sin. 15° = \cos. 45°$.

14. Show that $\cos. 22\frac{1}{2}° - \cos. 67\frac{1}{2} = \sqrt{\frac{2 - \sqrt{2}}{2}}$.

15. Show that $\cos. 67\frac{1}{2}° + \cos. 22\frac{1}{2} = \sqrt{\frac{2 + \sqrt{2}}{2}}$.

16. Show that $\sin. (90° + x) = \cos. x$.

17. Show that $\cos. (90° + x) = -\sin. x$.

18. Show that $\tan. (90° + x) = -\cot. x$.

19. Derive the sine, cosine, and tangent of an angle of 18°.

Suggestion. Let $x = 18°$; $5x = 90°$; $2x = 36°$; $3x = 54°$; therefore, $\sin. 2x = \cos. 3x$. Expand both terms and solve the equation.

20. Derive the sine, cosine, and tangent of an angle of 15°.

Suggestion. Let $2x = 30°$; then $x = 15°$. Use Art. 14 and equations of Art. 75.

21. Derive the sine and cosine of an angle of 7° 30′.
22. Derive the sine, cosine, and tangent of an angle of 22° 30′.
23. Derive the sine and cosine of an angle of 168° 45′.
24. Derive the sine and cosine of an angle of 176° 15′.
25. Derive the sine and tangent of an angle of 9°.
26. Derive the sine and tangent of an angle of 24°.
27. Derive the cosine and secant of an angle of 6°.
28. Derive the sine and tangent of an angle of 168°.
29. Prove $\tan.^2 x \sec.^2 x - \sec.^2 x + 1 = \tan.^4 x$.
30. Prove $\sec.^2 x + \dfrac{\sin. x}{\cos.^2 x} = \dfrac{1}{1 - \sin. x}$.
31. Prove $\sec. x \operatorname{cosec.} x \tan. x = \sec.^2 x$.
32. Prove $\dfrac{ab \sec.^2 \theta}{a^2 - b^2 \tan.^2 \theta} = \dfrac{ab}{a^2 \cos.^2 \theta - b^2 \sin.^2 \theta}$.
33. Prove $\dfrac{n \sec.^2 \theta}{1 + n^2 \tan.^2 \theta} = \dfrac{n}{\cos.^2 \theta + n^2 \sin.^2 \theta}$.
34. If $\tan. 3x \cos.^2 3x = \tan. x \cos.^2 x$, find x. *Ans.* $x = 22° 30′$.
35. If $a - b \tan. x = (a - b) \sqrt{1 + \tan.^2 x}$, find $\tan. x$.

Ans. $\dfrac{-ab \pm (a - b) \sqrt{2ab}}{a^2 - 2ab}$.

36. If $\cos.^3\phi = \sin. 2\phi \sin. \phi$, find ϕ.
37. If $1 - a^2 \cos.^2 x = 2 a^2 \sin.^2 x$, find $\sin. x$ and $\cos. x$.
38. If $a^2 \cos.^2 x \sin.^2 x + \sin.^2 x + a^2 \cos.^4 x = \cos.^2 x$, find $\sin. x$.

Ans. $\operatorname{Sin.} x = \pm \sqrt{\dfrac{1 - a^2}{2 - a^2}}$.

39. Given $\sin. x + \cos. 2x = \frac{7}{8}$; find x.
40. Given $\sec. x + \tan. x = \frac{5}{2}$; find x.
41. Given $\tan. x + \cot. x = 10$; find x.
42. Given $\sin. x + \cos. x = \frac{1}{3}$; find x.

CHAPTER VIII.

INVERSE TRIGONOMETRIC FUNCTIONS.—CIRCULAR MEASURE OF AN ANGLE.

ART. 76. If two quantities are so connected, that if one of them changes the other changes also, the second is said to be a *function* of the first.

Thus the sine, cosine, and tangent of any angle are functions of the angle, for when we change the angle the sine, cosine, tangent, etc., also change. In this sense, the *trigonometric ratios* of an angle are called its *trigonometric functions.*

If we consider the angle and any one of its functions as two *variables*, the angle will be the *independent* variable, and the function the *dependent* variable.

Thus in the equation $y = \sin. x$, when x changes y also changes, but the change of y *depends* upon the change of x.

Now, if we wish to consider the angle as a function of one of its trigonometric ratios—that is, if we wish to represent the angle as the *dependent*, and the ratio as the *independent* variable, we use a system of *inverse notation*, in which the negative exponent -1 is employed.

In the equation $y = \sin. x$, y is a function of x; but if we wish to express x as a function of y, we indicate the relation between y and x in the form:

$$x = \sin.^{-1} y,$$

and read the equation, x is an angle whose sine is y.

The relation between the angle and its other trigonometric ratios may be expressed in the same way.

Thus $x = \cos.^{-1} z$, $x = \tan.^{-1} w$, $z = \sec.^{-1} \frac{a}{b}$,

may be read; x equals an angle whose cosine is z; x equals an angle whose tangent is w; z equals an angle whose secant is $\frac{a}{b}$.

The expressions $\sin.^{-1} y$, $\cos.^{-1} z$, $\tan.^{-1} w$, $\sec.^{-1} \frac{a}{b}$, etc., may stand alone, and are called *inverse trigonometric functions.*

They may be read: an angle whose sine is y, an angle whose cosine is z, etc.

77. *In a circle whose radius is unity, to find the length of an arc of any number of degrees, minutes, and seconds.*

Denote the circumference of a circle by C, its radius by R; then $C = 2\pi R$ (Ch. Art. 40, V.).

Making radius equal to unity, we have $C = 2\pi$.

That is, 2π is the length of the circumference of a circle whose radius is unity. As there are 360° in the whole circumference, $1° = \frac{1}{360}$ of 2π, or $\frac{\pi}{180}$.

$$1' = \frac{1}{60} \text{ of } \frac{1}{360} \text{ of } 2\pi, \text{ or } \frac{\pi}{60 \times 180}.$$

$$1'' = \frac{1}{60} \text{ of } \frac{\pi}{60 \times 180}, \text{ or } \frac{\pi}{60 \times 60 \times 180}.$$

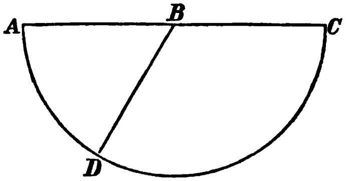
Denote the arc by a; then, generally, the length of an arc of n degrees in a circle whose radius is unity (n being any number, whole or fractional), is expressed by the equation,

$$a = \frac{n\pi}{180}.$$

78. In the ordinary measure of an angle and its intercepted arc by degrees, minutes, and seconds, the unit of an angle is the $\frac{1}{90}$th part of a right angle, and the unit of arc is the $\frac{1}{90}$th part of a quadrant (Ch. Art. 54, II.).

In the *circular measure* of an angle and its intercepted arc, *the unit of angle is an angle at the centre of a circle, subtended by an arc equal to the radius of the circle; and this arc is the unit of arc.*

79. *The angle at the centre of any circle subtended by an arc equal to the radius of the circle is a constant quantity.*



In the figure, suppose AD to be an arc equal in length to the radius, AB, of the semicircle ADC. Denote the radius by R; R will also represent the arc AD.

The length of the semicircle, ADC, equals πR (Ch. Art. 40, V.).

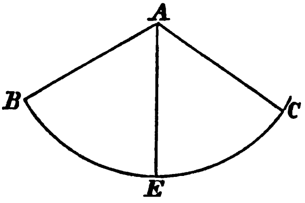
$$\frac{ABD}{2 \text{ right angles}} = \frac{AD}{ADC} \text{(Euc. 33, VI. Ch. 19, II.)}$$

$$= \frac{R}{\pi R} = \frac{1}{\pi};$$

therefore $A\,B\,D = \dfrac{2 \text{ right angles}}{\pi}$.

But $\dfrac{2 \text{ right angles}}{\pi}$ is independent of the radius of the circle, and, therefore, a constant quantity.

Therefore the angle $A\,B\,D$, or an angle at the centre of any circle subtended by an arc equal to the radius of the circle, is a constant quantity.



80. Let $B\,A\,C$ be any angle at the centre of a circle, and be denoted by A. Denote the intercepted arc, $B\,C$, by a. Let $B\,E$, an arc equal to the radius of the circle, be denoted by r, and let $B\,A\,E$, the subtended unit of angle, be denoted by u.

Then $\dfrac{A}{u} = \dfrac{a}{r}$ (Euc. 33, VI. Ch. 19, II.); but, as u is the unit of angle, we have

$$A = \frac{a}{r} \text{ (Ch. 20, II.).}$$

That is, the *circular measure of an angle is expressed by the ratio of the subtending arc to the radius in a circle described from the vertex of the angle as a centre.*

81. Four right angles at the centre of a circle are subtended by the whole circumference, or by $2\,\pi\,r$ (where r denotes the radius). Therefore, by the preceding article,

$$4 \text{ right angles} = \frac{2\pi r}{r} = 2\pi; \text{ that is,}$$

the circular measure of 4 right angles is 2π.

The circular measure of 2 right angles is $\frac{\pi r}{r} = \pi$; of 1 right angle is $\frac{\pi r}{2r} = \frac{\pi}{2}$; etc.

82. If, in the equation $A = \frac{a}{r}$, we let $r = 1$; that is, if we find the circular measure of an angle at the centre of a circle whose radius is unity, we derive the equation $A = a$; that is, *the circular measure of any angle, at the centre of a circle whose radius is unity, is equal to the length of the arc subtending that angle.*

83. Now, let A be the circular measure of any angle, and n the number of degrees, minutes, or seconds in the angle, n being integral or fractional; then, by Arts. 82 and 77,

$$A = \frac{n\pi}{180};$$

which is the general expression for the circular measure of any angle, when the degrees, minutes, or seconds which it contains are given.

In this system of measuring angles, an angle of:

4 right angles or an angle of 360° $= \frac{360}{180}\pi$ **or** 2π; **or**

4 right angles $= 2\pi = 6.2831854$;

2 right angles $= \frac{180}{180}\pi = \pi = 3.1415927$;

1 right angle $= \frac{90}{180}\pi = \frac{\pi}{2} = 1.5707963$.

$$\text{an angle of } 45° \text{ or } \tfrac{1}{2} \text{ right angle} = \frac{45}{180}\pi = \frac{\pi}{4} = .7853982;$$

$$\text{an angle of } 60° = \frac{60}{180}\pi = \frac{\pi}{3} = 1.0471976;$$

$$\text{an angle of } 30° = \frac{30}{180}\pi = \frac{\pi}{6} = .5235988;$$

$$\text{an angle of } 1° = \frac{\pi}{180} = .01745329;$$

$$\text{an angle of } 1' = \frac{\pi}{60 \times 180} = .000290888;$$

$$\text{an angle of } 1'' = \frac{\pi}{60 \times 60 \times 180} = .00000485.$$

84. *To find the number of degrees, minutes, and seconds in the unit angle of circular measure.*

In Art. 79 the unit angle is shown to be equal to $\frac{2 \text{ right angles}}{\pi}$.

2 right angles are measured by 180°:

Therefore the unit angle =

$$\frac{180°}{\pi} = \frac{180°}{3.1415927} = 57°.295779 = 57°17'44.8.''$$

85. *To change the measure of an angle in degrees, minutes, or seconds, to the equivalent circular measure; and conversely, to change the circular measure of an angle to its measure in degrees, minutes, or seconds.*

Let A be the circular measure of any angle, *n* the number of degrees, minutes, or seconds which it contains, *n* being any number, integral or fractional. According to Art. 83,

$$(1)\ \mathrm{A} = \frac{n\,\pi}{180};\ \text{therefore,}$$

$$(2)\ n = \mathrm{A}\,\frac{180^\circ}{\pi} = \mathrm{A} \times 57^\circ.295779.$$

To change degree measure into circular measure we use equation (1); to change circular measure into degree measure, we use equation (2).

1. Show that $\tan.^{-1} 1 + \tan.^{-1} (-1) = \pi$.

2. Show that $\sin.^{-1} \frac{1}{2} + \sin.^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{2}$.

3. If $\sin.^{-1} x + \sin.^{-1} \frac{2}{3} x = \frac{\pi}{3}$, find x. *Ans.* $x = \frac{3\sqrt{3}}{2\sqrt{19}}$.

4. Prove $\sin.^{-1} \frac{33}{65} = \sin.^{-1} \frac{4}{5} - \sin.^{-1} \frac{5}{13}$.

5. Prove $\sin.^{-1} \left(\frac{2\sqrt{15} + 7\sqrt{5}}{24}\right) = \sin.^{-1} \frac{2}{3} + \sin.^{-1} \frac{7}{8}$.

6. Prove $\sin.^{-1} \left(\frac{3\sqrt{5} + 2\sqrt{7}}{12}\right) = \sin.^{-1} \frac{3}{4} + \sin.^{-1} \frac{2}{3}$.

7. Prove $\tan.^{-1} \frac{29}{14} = \tan.^{-1} \frac{6}{5} + \tan.^{-1} \frac{1}{4}$.

8. Prove $\tan.^{-1} \frac{16}{17} = \tan.^{-1} \frac{13}{7} - \tan.^{-1} \frac{1}{3}$.

9. Find the number of degrees, minutes, and seconds, in an angle whose circular measure is $\frac{\pi}{14}$. *Ans.* 12° 51′ 25.7″.

10. Find the number of degrees, minutes, and seconds, in an angle whose circular measure is $\frac{2\pi}{13}$.

11. Find the number of degrees, minutes, and seconds, in an angle whose circular measure is $\frac{6}{5}$, that is, $\frac{6}{5}$ of the unit angle.

Ans. 68° 45′ 17.77″.

12. Find the number of degrees, minutes, and seconds, in an angle of which the circular measure is .763.

13. What is the circular measure of an angle of 11° 15′ ? *Ans.* $\frac{\pi}{16}$.

14. What is the circular measure of an angle of 1° 30′ ?

15. What is the circular measure of an angle of 7° 30′ ?

16. What is the number of degrees, minutes, and seconds, in an arc 40 feet long, in a circle whose radius is 35 feet ?

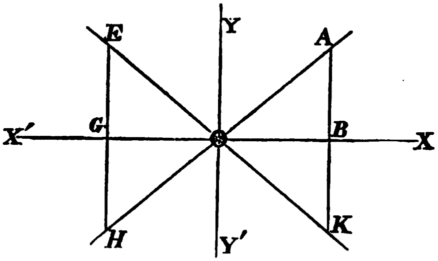
Solution. Degrees in arc = degrees in angle = unit $\times \frac{\text{arc}}{\text{radius}} = \frac{180°}{\pi} \times \frac{40}{35} = 57°.295779 \times \frac{8}{7} = 65° \, 28' \, 51.2''$.

17. What is the number of degrees, minutes, and seconds in an angle subtended by an arc of 25 feet in a circle whose radius is 15 feet ?

CHAPTER IX.

ANGLES HAVING CORRESPONDING TRIGONOMETRIC FUNCTIONS OF THE SAME NUMERICAL VALUE.—TABLE OF IMPORTANT TRIGONOMETRIC FUNCTIONS.—ANGLES GREATER THAN FOUR RIGHT ANGLES.—POSITIVE AND NEGATIVE ANGLES.

ART. 86. *Angles having corresponding trigonometric functions of the same numerical value.*



In the figure, let XX' and YY', two lines at right angles at O, be the *initial* lines (Art. 41).

Beginning with XOY, and going from right to left, call XOY the *first quadrant*, YOX' the *second quadrant*, $X'OY'$ the *third quadrant*, and $Y'OX$ the *fourth quadrant.*

Let two lines, AH and EK, be drawn making equal angles with XX' and intersecting XX' and one another at the same point, O. Then will the angles XOA, EOX', $X'OH$, and KOX be equal to one another.

If, now, we take the points A, E, H, and K, equally distant from O, and join the points A and K by the straight line AK, and the points E and H by the straight line EH, we shall have four equal right-angled triangles, BOA, EOG, GOH, and KOB (Euc. 4, I. Ch. 20, I).

Besides these four right-angled triangles, there can be no others equal to them, having their bases and perpendiculars in the same relative position, with respect to the initial lines and the point O.

Reckoning the angles from OX (from right to left) through the four quadrants, there will be, therefore, only *four* angles, having equal *triangles of reference* (Art. 3).

The triangle of reference for an *acute* angle will be in the *first quadrant;* for an *obtuse* angle, in the *second quadrant;* for an angle *greater than two right angles* but *less than three right angles*, in the *third quadrant;* for an *angle greater than three right angles but less than four right angles*, in the *fourth quadrant.*

The *numerical values of the six trigonometric functions*, sine, cosine, tangent, cotangent, secant, and cosecant, *of these four angles will be the same* (Euc. 4, VI. Ch. 4, III).

Thus the angles BOA, BOE, and the salient angles BOH and BOK, will have equal triangles of reference, and, consequently, equal numerical values for the corresponding trigonometric functions, sine, cosine, tangent, cotangent, secant, and cosecant.

87. *To find expressions for the four angles whose trigonometric functions have the same numerical value.*

In the figure, denote the angle BOA by a, then each of the equal angles, EOG, GOH, and KOB, will also be denoted by a.

BOA, BOE, the salient angles BOH and BOK, are the angles which have the same numerical values for their corresponding trigonometric functions.

$BOA = a.$

$BOE = 2$ right angles $- EOG = 180° - a = \pi - a.$

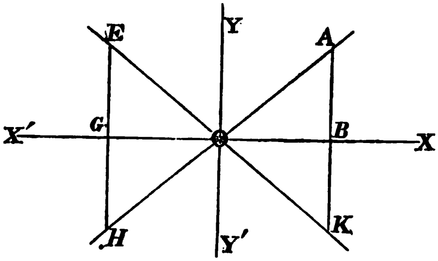
$BOH = 2$ right angles $+ GOH = 180° + a = \pi + a.$

$BOK = 4$ right angles $- KOB = 360° - a$
$= 2\pi - a.$

Denoting then any acute angle by a, the four angles, which have the same numerical values for their corresponding trigonometric functions, are:

$$a,\ 180° - a,\ 180° + a,\ 360° - a;\ \text{or},$$
$$a,\ \pi - a,\ \pi + a,\ 2\pi - a.$$

88. *To find the signs of the trigonometric functions of angles having equal triangles of reference.*



In the figure, let the construction be the same as in Art. 86.

In the first quadrant, the *perpendicular* and *base* of the triangle of reference, AOB, are *positive* (Art. 41);

therefore, as the hypotenuse is positive, *all the trigonometric functions of an angle in the first quadrant are positive*, all the ratios having like signs in both their terms.

In the second quadrant, YOX', the *perpendicular* is always *positive*, while the *base* is always *negative;* therefore, where the triangle of reference is in the second quadrant, the trigonometric functions of the angle in which the *perpendicular* occurs with the *hypotenuse* are *positive*, both terms of the ratios having like signs; and the trigonometric functions in which the *base* occurs are *negative*, as the two terms of these ratios have opposite signs.

That is, of an obtuse angle the *sine* and *cosecant* are *positive*, but the *cosine*, *tangent*, *cotangent*, and *secant are negative.*

In the third quadrant, $X'OY'$, the *perpendicular* and *base* are both *negative;* therefore, where the triangle of reference is in the *third* quadrant, the trigonometric functions of the angle in which the *perpendicular* or *base* occur with the *hypotenuse* are *negative;* but those in which the *base* and *perpendicular* occur together are *positive.*

That is, of an angle greater than two right angles and less than three right angles, or greater than π and less than $\frac{3\pi}{2}$, the *sine*, *cosine*, *secant*, and *cosecant* are *negative*, but the *tangent* and *cotangent* are *positive.*

In the fourth quadrant, $Y'OX$, the *perpendicular* is always *negative*, while the *base* is *positive;* therefore, when the triangle of reference is in the *fourth* quadrant, the trigonometric functions of the angle in which the *perpendicular* occurs are *negative;* and the trigonometric functions in which the *base* occurs with the *hypotenuse* are *positive.*

That is, of an angle greater than three right angles and less than four right angles, or greater than $\frac{3\pi}{2}$ and less than 2π, the *sine, tangent, cotangent,* and *cosecant* are *negative,* but the *cosine* and *secant* are *positive.*

The versed sine is always positive.

The following table will show the results of the preceding article:

ANGLE.	SIN.	TAN.	SEC.	COSIN.	COTAN.	COSEC.	VERSIN.
> 0 and $< \frac{\pi}{2}$	+	+	+	+	+	+	+
$> \frac{\pi}{2}$ and $< \pi$	+	−	−	−	−	+	+
$> \pi$ and $< \frac{3\pi}{2}$	−	+	−	−	+	−	+
$> \frac{3\pi}{2}$ and $< 2\pi$	−	−	+	+	−	−	+

89. *Particular angles having equal triangles of reference.*

(1) In the figure of Art. 86, let $A\,O\,B$ or a, equal 0; then the expressions, a, $\pi - a$, $\pi + a$, $2\pi - a$ (Art. 87), reduce to 0, π, and 2π;

Therefore the corresponding trigonometric functions of 0, π, and 2π, are equal in *numerical* value (Art. 39).

As the perpendicular of the triangle of reference disappears at 0, at π, and at 2π, *the signs* of the trigonometric ratios, in which the *perpendicular* occurs, will be *undetermined;* but, as the base remains, the signs of the ratios in which the *base* occurs with the hypotenuse will be *determined* by the sign of the base, and will therefore be *positive* at 0 and 2π, but *negative* at π.

(2) Let $A\,O\,B$ or a equal $\frac{\pi}{2}$; then the expressions a, $\pi - a$, $\pi + a$, $2\pi - a$, reduce to $\frac{\pi}{2}$ and $\frac{3\pi}{2}$; therefore the corresponding trigonometric functions of $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are the same in *numerical* value (Arts. 35, 36, 37, 38).

As the base of the triangle of reference disappears at $\frac{\pi}{2}$ and at $\frac{3\pi}{2}$, the *signs* of the ratios in which the *base* occurs will be *undetermined;* but as the perpendicular remains, the signs of the ratios in which the *perpendiculr* occurs with the *hypotenuse*, will be determined by the sign of the perpendicular, and will therefore be *positive* at $\frac{\pi}{2}$, but *negative* at $\frac{3\pi}{2}$.

Let $a = \frac{\pi}{6}$; then, by Art. 87,

(3) $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, will have their corresponding trigonometric functions the same in numerical value (Art. 14).

Let $a = \frac{\pi}{4}$; then,

(4) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, will have their corresponding trigonometric functions the same in numerical value (Art. 15).

Let $a = \frac{\pi}{3}$; then,

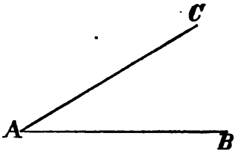
(5) $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$, will have their corresponding trigonometric functions the same in numerical value (Art. 16).

The signs of the functions (3), (4), and (5) will be determined by Art. 88.

(90) The following table will show the results of the preceding article:

Angle	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Sine..........	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
Tangent.....	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
Secant.......	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cosine.......	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Cotangent...	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	∞
Cosecant.....	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	∞

(91). *Angles greater than four right angles.*



If the line $A\ C$ be conceived as first coinciding with $A\ B$, and then to revolve about A, from right to left, till it again coincide with $A\ B$, it will pass through four right angles. If, after coinciding with $A\ B$, it still continues to revolve, from right to left, till it takes the position of $A\ C$ (in the figure), it is said to make with $A\ B$ *an angle greater than four right angles.*

Let $C A B$ be denoted by a. Then the angle, greater than four right angles, which $C A$ makes with $A\ B$, will be expressed by $2\pi + a$.

If $C A$ be revolved twice through four right angles, and then continue to revolve till it takes the position $A\ C$; the angle, greater than eight right angles, which $A\ C$ makes with $A\ B$, will be expressed by $4\pi + a$.

Generally an angle greater than four right angles may be expressed by $2n\pi + a$, where n is any integral, and a is the excess of the angle over $2n\pi$.

92. *Trigonometric functions of angles greater than four right angles.*

It is evident, from the figure, that the trigonometric functions of the angle $2n\pi + a$ will be the same as those of a, as they will have the same triangle of reference.

Therefore, to find the trigonometric ratios of an angle greater than four right angles, we subtract from the angle four right angles, or some multiple of four right angles, and take the trigonometric ratios of the remaining angle.

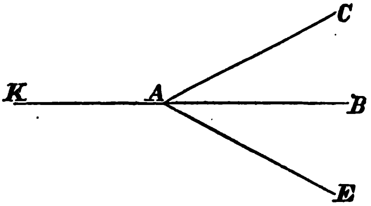
Thus $\sin.\ (2n\pi + a) = \sin.\ a$
$\tan.\ (2n\pi + a) = \tan.\ a$
$\sec.\ (2n\pi + a) = \sec.\ a$
$\cos.\ (2n\pi + a) = \cos.\ a$
$\cot.\ (2n\pi + a) = \cot.\ a$
$\text{cosec.}\ (2n\pi + a) = \text{cosec.}\ a$
$\text{versin.}\ (2n\pi + a) = \text{versin.}\ a$

93. It has been shown that an acute angle, an obtuse angle, an angle greater than two right angles, an angle greater than three right angles, an angle greater than four right angles, all have equal triangles of reference (Arts. 86, 91).

Therefore the relations (established for acute and obtuse angles, in Chapter VII.), of trigonometrical functions of the same angle to each other, and the relations of the trigonometrical functions of the sum or differance of two angles to the functions of the angles, and the relations following these, may also be considered as established for any angle whatever, as all these relations depend upon the triangles of reference.

94. *Positive and negative angles.*

If a horizontal line, *K B*, be taken as an initial line, and a straight line, as *A C*, be made to coincide with *A B*, and then to revolve about *A from right to left*, assuming various positions, the angles which it then makes with *A B* are *positive* angles.

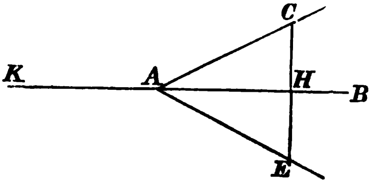


If the line *A C* be revolved about *A* in a contrary direction (i. e., as the hand of a watch), the angles which it then makes with *A B* are *negative angles.*

Thus, in the figure, $B\,A\,C$ is a *positive* angle; $B\,A\,E$ is a *negative* angle.

95. *Relations between the trigonometric functions of positive angles, and of equal negative angles.*

In the figure, let $K\,B$ be the initial line; $B\,A\,C$ be a positive angle; and $B\,A\,E$ an equal negative angle. Denote the angle $B\,A\,C$ by a; then $B\,A\,E$ will be denoted by $-a$.



From C, any point in $A\,C$, draw a perpendicular to $A\,B$, and produce it to meet $A\,E$ in E.

The two triangles of reference, $C\,A\,H$ and $H\,A\,E$, are equal in all respects, except that the sign of $E\,H$ is negative, while the sign of $C\,H$ is positive; therefore the trigonometric functions of $-a$ will be numerically equal to those of a, but the *signs* of the functions of the two angles in which the perpendicular occurs will be unlike, while the signs of the functions in which the base occurs with the hypotenuse will be the same (Art. 41).

Thus, $$\sin.(-a)=\frac{E\,H}{A\,E}=\frac{-C\,H}{C\,A}=-\sin.\ a;$$

$$\tan.(-a)=\frac{E\,H}{A\,H}=\frac{-C\,H}{A\,H}=-\tan.\ a;$$

$$\sec.(-a)=\frac{A\,E}{A\,H}=\frac{A\,C}{A\,H}=\sec.\ a;$$

$$\cos.(-a)=\frac{A\,H}{A\,E}=\frac{A\,H}{A\,C}=\cos.\ a;$$

$$\cot.(-a)=\frac{A\,H}{E\,H}=\frac{A\,H}{-C\,H}=-\cot.\ a;$$

$$\text{Cosec.}\,(-a) = \frac{AE}{EH} = \frac{AC}{-CH} = -\,\text{cosec.}\,a\,;$$

$$\text{Versin.}\,(-a) = 1 - \frac{AH}{AE} = 1 - \frac{AH}{AC} = \text{versin.}\,a.$$

Example 1. Find the sine and cosine of $\frac{9\pi}{8}$, and of $\frac{15\pi}{8}$.

2. Find the sine and cosine of α, $\pi - \alpha$, $\pi + \alpha$, and $2\pi - \alpha$, where $\alpha = 15°$.

3. What angles are expressed in degrees by $\pi(n \pm \frac{1}{6})$, n being 0 or any integral number? Write the sines of these angles.

4. Write the cosines for the series of angles expressed by $\pi(n \pm \frac{1}{3})$, n being 0, or any integral.

5. Write the tangents of the series of angles $\pi(n \pm \frac{1}{4})$, n being 0, or any integral.

6. Write the trigonometric ratios for the angles $\frac{(2n+1)\pi}{2}$, n being 0, or any integral.

7. Prove $\dfrac{\sec.^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\tan.\left(\frac{\pi}{4} + \frac{x}{2}\right)} = \dfrac{1}{\cos.\,x}$.

8. Prove $1 + \dfrac{\sin.\left(\frac{\pi}{4} - x\right)}{\cos.\left(\frac{\pi}{4} - x\right)} = \dfrac{2}{1 + \tan.\,x}$.

9. Prove that $x + \frac{1}{2}y = \pi$, if $\dfrac{\sin.\,y}{1 - \cos.\,y} + \dfrac{\cos.\,x}{\sin.\,x} = 0$.

10. Find two circular measures of θ which will satisfy the equation $\cos.\,\theta + \cos.^2\theta = \sin.^2\theta$. *Ans.* $\theta = \frac{\pi}{3}$ and $\theta = \pi$.

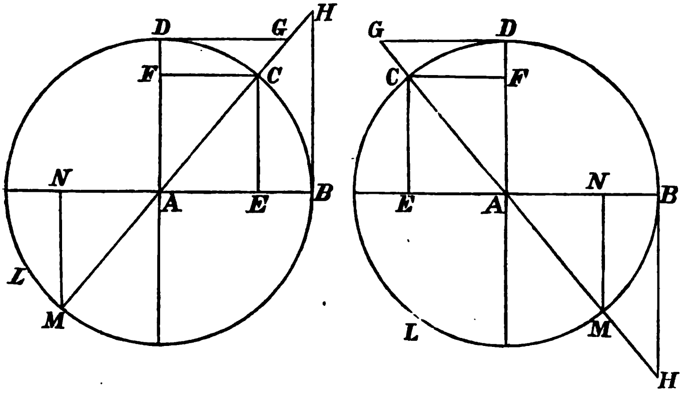
11. Find two circular measures of θ which will satisfy the equation $2\cos.\,\theta\sin.\,2\theta - 2\cos.\,2\theta = 2\cos.\,2\theta\sin.\,\theta$. *Ans.* $\theta = \frac{3\pi}{2}$ and $\frac{\pi}{6}$.

12. Find one circular measure of ϕ which will satisfy the equation $3\sin.^2\phi + 2\sin.\,\phi = 1$.

CHAPTER X.

TRIGONOMETRIC FUNCTIONS REPRESENTED BY LINES.—OUTLINE OF A METHOD OF CONSTRUCTING TRIGONOMETRIC TABLES.

Art. 96. The trigonometric functions of an angle are sometimes represented by lines having certain positions in, or with respect to, a circle described from the vertex of the angle as a centre, and with a radius regarded as a unit of length.



In the figure, let $B\,A\,C$ be any angle.

From the vertex A, as a centre, with any unit of length as a radius, describe the circle $B\,C\,L$, intersecting the sides of the angle at B and C. At A draw $D\,A$ at right angles to $A\,B$. From C draw $C\,E$ at right angles to

A B or *A B* produced. From *B* draw the tangent *B H* meeting *A C* produced at *H*. Also, from *C* draw *C F* perpendicular to *A D* and from *D* draw the tangent *D G*.

Let *A C*, produced through the centre, meet the circumference at *M*, and the tangent *B H* at *H*.

From *M* draw *M N* perpendicular to *A B*, or *A B* produced.

The *sine* of an angle (or its subtending arc), is the perpendicular drawn from one extremity of the arc to the radius, or radius produced, passing through the other extremity of the arc.

Thus, sin. *B A C* or sin. *B C* = *C E*.

The trigonometric *tangent* of an angle (or its subtending arc), is the part of the geometric tangent at one extremity of the arc, terminated by the point of contact and the produced radius passing through the other extremity of the arc.

Thus, tan. *B A C* or tan. *B C* = *B H*.

The *secant* of an angle (or its subtending arc), is the part of the produced radius between the centre and the tangent.

Thus, sec. *B A C* or sec. *B C* = *A H*.

The *cosine*, *cotangent*, and *cosecant* of an angle, (or its subtending arc), are the sine, tangent, and secant, respectively, of the complement of the angle or arc.

Thus: cos. *B A C*, or cos. *B C* = sin. *C A D*
= sin. *C D* = *C F*;

cotan. *B A C* or cot. *B C* = tan. *C A D* = tan. *C D*
= *D G*;

cosec. *B A C* or cosec. *B C* = sec. *C A D* = sec. *C D*
= *A G*.

The versed sine of an angle (or arc) is the distance between the foot of the sine (drawn from one extemity of the arc) and the other extremity of the arc, measured on the radius, or radius produced.

Thus, versin. $B\,A\,C$, or versin. $B\,C = B\,E$.

It is evident from the figure that the sines and cosines of all angles not 0, or not multiples of a right angle, are less than the radius, that is, less than 1 (1 standing for the unit of length) (Euc. 15, III. Ch. 3, II.).

By bisecting the arc of the quadrant $B\,D$, and drawing tangents, it can be shown that a tangent may be less than radius, equal to radius, or greater than radius; that is, less than 1, equal to 1, or greater than 1 (Euc. 6 and 19, I. Ch. 27 and 28, I.).

It is also evident from the figure, that the secants of all angles not 0, or not equal to two or to four right angles, are greater than radius, that is, greater than 1.

At 0° the sine is 0, the tangent is 0, the secant 1, and the cosine 1.

At 90° the sine is 1, the tangent ∞, the secant ∞, and the cosine 0.

97. Lines from *above*, perpendicular upon $A\,B$, or $A\,B$ produced are *positive;* and lines from *below*, perpendicular to $A\,B$ are *negative.*

Perpendicular lines proceeding from $A\,D$, or $A\,D$ produced, to the *right* are *positive*, while those proceeding to the *left* are *negative.*

Also, the radius and lines measured on the radius or radius produced, *measured from* the centre *to* or *through* one *extremity* of the arc are positive; but lines measured on the radius produced through the centre in the *opposite direction* from the *extremity* of the *arc* are *negative.*

Thus, all the lines representing the trigonometric functions of an acute angle are positive.

Of the lines representing the trigonometric functions of an obtuse angle, the sine and cosecant are positive; but the cosine, tangent, cotangent, and secant are negative.

That is, (in the right-hand figure, page 105), CE and AG are positive; AE, BH, DG, and AH are negative.

If (in the left-hand figure) we have an angle, BAM (salient), greater than two right angles:

its sine MN, cosine AN, secant AH, and cosecant AG are negative; but the tangent BH and cotangent DG are positive.

If (in the right-hand figure) we have a salient angle, BAM, greater than three right angles:

its sine MN, tangent BH, cotangent DG, and cosecant AG are negative; but the cosine AN and secant AH are positive.

The versed sine in all cases is positive.

The results of this article will, on comparison, be seen to agree with the results of Art. 88.

98. From the similar triangles BAH and EAC (*see* figure on page 105),

$$BH : CE :: BA : AE \text{(or } CF\text{)};$$

therefore, $BH = \dfrac{BA \times CE}{CF}$; or, denoting the angle BAC by A, and the radius by R, $\tan. A = \dfrac{R \sin. A}{\cos. A}$;

but, as $R = 1$, $\tan. A = \dfrac{\sin. A}{\cos. A}$.

In a similar manner $AH : AC :: AB : AE$;

therefore $AH = \sec. A = \dfrac{R^2}{\cos. A} = \dfrac{1}{\cos. A}$.

From similar triangles, ACF and AGD,

$$AG : AC :: AD : AF (= CE);$$

therefore, $AG = \operatorname{cosec.} A = \dfrac{R^2}{\sin. A} = \dfrac{1}{\sin. A}$.

In a similar manner:—

$$DG : FC :: AD : AF (= CE);$$

therefore, $DG = \cot. A = \dfrac{R \cos. A}{\sin. A} = \dfrac{\cos. A}{\sin. A}$.

Again, from similar triangles, ADG and ABH,

$$DG : AB :: DA : BH;$$

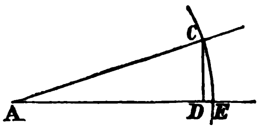
therefore, $DG = \cot. A = \dfrac{R^2}{\tan. A} = \dfrac{1}{\tan. A}$.

These results have already been obtained by the method of ratios (Art. 64).

99. In Arts. 14 and 15 we have obtained the trigonometric functions of an angle of 30° and of 45° respectively. By means of the solution of Example 19 at the end of Chapter VII., we can find the trigonometric functions of an angle of 18°. By Art. 75 we can then obtain, from these functions, the functions of an angle of 15°, $22\frac{1}{2}$°, and 9°; and then the functions of an angle of 7° 30′, 11° 15′, 4° 30′; and then the functions of the halves of these angles, and so on. Also by means of Arts. 67, 68, and 72, we can obtain the functions of the sums and differences of the angles thus found.

By these methods, however, we can ascertain the trigonometric functions of only a part of the angles of the first quadrant.

By the method given in outline in the following article, we can ascertain the trigonometric functions of any angle of the quadrant.



100. *Outline of a method of constructing tables of trigonometric functions of angles of the first quadrant.*

Let A be a small acute angle; CAD its triangle of reference.

From A as a centre with a radius AC describe the arc CE, intersecting the sides of the angle in C and E.

If DAC were one of a number of equal angles at the centre of a circle, CD would be half the side of a regular polygon inscribed in the circle. By increasing the number of sides of the polygon, that is, by decreasing the size of the angle at the centre, CD would very nearly equal the arc CE (Ch. 14, V.). Consequently, if the angle A were small enough, $\frac{CD}{CA}$ would approximately equal $\frac{CE}{CA}$. But $\frac{CD}{CA} = \sin. A$ ((1) Art. 4), and $\frac{CE}{CA} =$ the circular measure of A (Art. 80); therefore, of *a very small angle the sine approximately equals the circular measure.*

In the construction of trigonometric tables, the angle A is taken as very small, and the sine of the angle is taken as *equal* to its circular measure. The angle is sometimes taken as 1′ and sometimes as 10″.

Suppose, then, A equal to $1'$, and that its sine is taken as equal to its circular measure; then:

Sin $1' = .00029089$ (Art. 83).

Cos. $1' = \sqrt{1 - \sin.^2 1'} = \sqrt{(1 + \sin. 1')(1 - \sin. 1')}$
$= \sqrt{1.00029089 \times .99970911} = .999999957$.

Now by equations (a) and (c) of Art. 69,

Sin. $(x + y) = 2 \sin. x \cos. y - \sin. (x - y)$.

Cos. $(x + y) = 2 \cos. x \cos. y - \cos. (x - y)$.

In these equations let $y = 1'$, and let x be in succession $1'$, $2'$, $3'$, etc., and we shall have for the sines

(1) $\text{Sin.}2' = 2 \sin.1' \cos.1'$;

(2) $\text{Sin.}3' = 2 \sin. 2' \cos. 1' - \sin.1'$;

(3) $\text{Sin.}4' = 2 \sin.3' \cos.1' - \sin.2'$, etc.;

and for the cosines

(4) $\text{Cos.}2' = 2 \cos.^2 1' - 1$;

(5) $\text{Cos.}3' = 2 \cos.2' \cos.1' - \cos.1'$;

(6) $\text{Cos.}4' = 2 \cos.3' \cos.1' - \cos.2'$; etc.

By substituting in equation (1) for sin. $1'$, its value, and for cos. $1'$ its value, we obtain sin. $2'$; then, by substitution of values in (2) we obtain sin. $3'$; and so on.

By substituting in equation (4) for cos. $1'$ its value, we obtain cos. $2'$; and then by substitution of values in (5) we obtain cos. $3'$; and so on.

Thus, starting with the sine and the cosine of $1'$, we find the sine and cosine of $2'$, of $3'$, of $4'$, etc.

The tangents, cotangents, secants, and cosecants are derived from the sines and cosines by means of the equations of Art. 64.

Thus, tan. $2' = \dfrac{\sin. 2'}{\cos. 2'}$; sec. $2' = \dfrac{1}{\cos. 2'}$; etc.

It is not necessary to obtain the functions of all the angles of the quadrant by the above method. For instance, suppose we have obtained the functions of angles up to 45°, at intervals of 1′. Then, denoting by x the excess of any angle over 45°, we have—

$$\sin.\ (45^\circ + x) = \cos.\ (45^\circ - x);$$
$$\cos.\ (45^\circ + x) = \sin.\ (45^\circ - x);$$
$$\tan.\ (45^\circ + x) = \cot.\ (45^\circ - x)\ \text{(Art. 5)};$$

since $45^\circ + x$ is the complement of $45^\circ - x$.

Let the angle, whose functions are required, be 47° 35′. Then

$$x = 2^\circ\ 35', \text{ and}$$
$$\sin.\ 47^\circ\ 35' = \cos.\ 42^\circ\ 25';$$
$$\cos.\ 47^\circ\ 35' = \sin.\ 42^\circ\ 25';$$
$$\tan.\ 47^\circ\ 35' = \cot.\ 42^\circ\ 25'; \text{ etc.}$$

The right-hand members of these equations are supposed to be already obtained. Therefore we shall know their equals, sin. 47° 35′, etc.

The results obtained by the methods of Art. 99 may also be used to test the accuracy of the work by the method of this article.

CHAPTER XI.

DEFINITIONS.—THEOREMS OF RIGHT-ANGLED TRIANGLES.

ART. 101. SPHERICAL TRIGONOMETRY teaches the methods of finding the unknown parts of *triedral* angles from certain known parts, by means of the solution of *spherical triangles.*

102. A *spherical triangle* is formed by the intersection of the *planes* of a triedral angle with the *surface* of the sphere, the *vertex* of the triedral angle being at the *centre* of the sphere. The *sides* of the spherical triangle are arcs of great circles (Ch. Art. 26, VIII.), which measure the face angles of the triedral angle (Ch. Art. 53, II.). The *angles* of the spherical triangle are equal to the diedral angles of the triedral angle (Ch. 16, VIII.).

103. The face angles of a triedral angle are assumed to be each less than two right angles. Therefore *each side* of a spherical triangle will always be assumed as *less* than 180°. *Each angle* of a spherical triangle is *less* than 180°.

(*a*) The trigonometric functions of a *side* of a triangle are the trigonometric functions of the angle measured by the side.

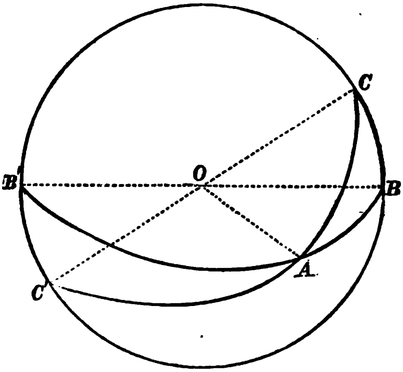
104. A *right triedral* angle is a triedral angle one of whose diedral angles is a *right diedral angle* (Ch. Art. 43, VI.).

A RIGHT-ANGLED spherical triangle is a spherical triangle one of whose angles is a right angle. It is formed on the surface of a sphere by the planes of a right triedral angle, whose vertex is at the centre of the sphere.

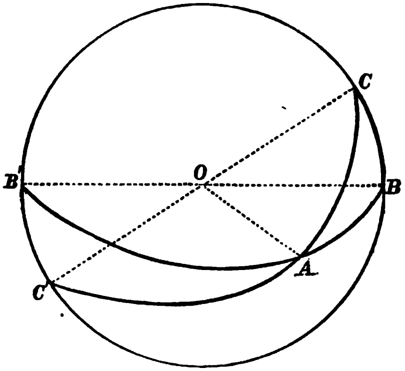
The *hypotenuse* is the side *opposite* the right angle.

105. A *quadrantal* triangle is a spherical triangle one of whose sides is a *quadrant.* It is formed on the surface of a sphere by the planes of a triedral angle (whose vertex is at the centre of the sphere), one of whose face angles is a right angle.

106. *In a right-angled spherical triangle, not a quadrantal triangle, the three sides are each less than* 90°; or, of the three sides, *one is less than* 90°, *and the other two are each greater than* 90°.



Let $OABC$ be a right triedral angle, of which each of the face angles is less than 90°. The planes of these face angles will form on the surface of the sphere a triangle ABC, each of whose sides is less than 90° (Art. 102). Let the triangle be represented on the surface of a hemisphere of which the base is the circle $BCB'C'$. Let the planes of the face angles, AOC and AOB, produced intersect the surface of this hemisphere in the arcs CAC', BAB' (which are semicircles (Ch. Art. 32, VIII.), and let the face BOC coincide with the base of the hemisphere.



By hypothesis the triangle ABC has its three sides each less than 90°.

In the triangle BAC', BA is *less* than 90°; BC' and AC' are each the supplement of an arc less than 90°, and are, therefore, each *greater* than 90°.

In the triangle $B'AC$, AC is *less* than 90°; $B'C$ and $B'A$ are each greater than 90°, being each the supplement of an arc less than 90°. $B'OC' = BOC$; therefore the arc $B'C'$ = the arc BC, and is less than 90°. Consequently in the triangle $B'AC'$, one side, $B'C'$, is less than 90°, while the other two sides, $B'A$ and $C'A$, are each greater than 90°, being each the supplement of an arc which is less than 90°.

The triangles ABC, BAC', CAB', and $B'AC'$ are the only, four kinds of triangles, not quadrantal, into which the surface of the hemisphere can be divided. The four triangles, on the surface of the other hemisphere, formed by producing the planes AOC and AOB to meet the surface of that hemisphere, will be symmetrical to these (Ch. Art. 63, VIII.), and will therefore have their sides either each less than 90°, or one side less than 90°, and the other two sides greater than 90°.

The principle stated therefore holds true for the whole surface of the sphere.

107. Two angles or two arcs, or an angle and an arc, are said to be of the *same species* when both are less than 90°, or both are greater than 90°.

108. *In a right-angled spherical triangle*, a *side* about the right angle *and its opposite angle* are always *of the same species.*

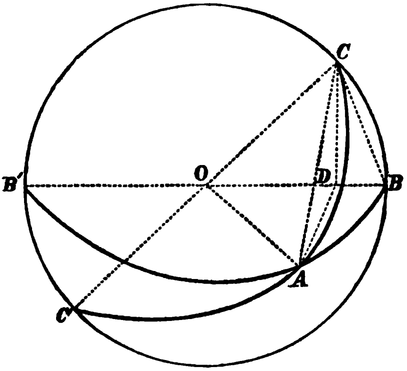
Let a triangle ABC be constructed on the surface of a hemisphere having its sides each less than 90°, and let the figure be completed as in Art. 106. Then the hypotenuse will be greater than either of the other sides.

For from C draw, in the plane BOC, a straight line, CD, perpendicular to the radius OB. CD will be perpendicular to the plane AOB (Ch. 18, VI.), and therefore perpendicular to the straight line AD, in that plane, drawn from D to A (Ch. Art. 6, VI.).

Now $OD + DA > OA$ (Euc. 20, I. Ch. 17, I.); therefore $OD + DA > OB$, and taking away OD

$$DA > DB;$$

therefore the chord CA is greater than the chord CB (Euc. 47, I. Ch. 4, VI.), and the arc CA is greater than the arc CB. Since the arc CA is greater than the arc CB, the angle CBA is greater than the angle CAB (Ch. 26, VIII.); but CBA is an angle of 90°; therefore the angle CAB is less than 90°; but the side CB is by hypothesis less than 90°; consequently the side CB and its opposite angle are of the *same species.*



In a similar manner it can be proved that AB and the angle ACB are of the same species.

In the triangle BAC', BA and $BC'A$ are both *less* than 90°; BC' is the supplement of BC, which is less than 90°, and is therefore greater than 90°; but its opposite angle BAC' is also greater than 90° being the supplement of BAC, which has just been proved to be less than 90°.

In the triangle $B'AC$, the two sides $B'A$, $B'C$, and their opposite angles $B'CA$, $B'AC$ are all *greater* than 90°, being the supplements of parts of the triangle BAC, which are all *less* than 90°.

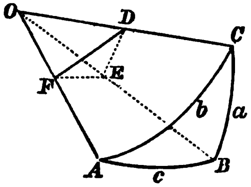
In the triangle $B'AC'$, $B'C'$ and its opposite angle

A are both *less* than 90°, while $B'A$ and the opposite angle $B'C'A$ are both greater than 90°.

The principle of the article holds true for the triangles on the surface of the hemisphere represented by the figure; it also holds true for the triangles symmetrical to these on the surface of the other hemisphere, and therefore holds true for the surface of the whole sphere.

If the triangle had two right angles, the sides opposite these would be quadrants (Ch. Art. 89, VIII.), but the remaining side and its opposite angle would be of the same species (Ch. 16, VIII.).

109. *In a right-angled spherical triangle, the sine of either of the sides about the right angle is equal to the product of the sine of the hypotenuse by the sine of the angle opposite the side.*[*]



Let ABC be a right-angled triangle on the surface of a sphere whose centre is the vertex, O, of the triedral angle, which forms the triangle by the intersection of its planes with the surface. Let B be a right angle, then OB is the edge of a right diedral angle (Art. 104).

Then we are to prove sin. a = sin. b sin. A.

From any point F in OA draw FD in the plane AOC, and FE in the plane AOB, both straight lines at right angles to OA. Let FD meet OC at D, and let FE meet OB at E. Draw the straight line DE.

OA is perpendicular to the plane DFE (Ch. 5, VI.).

The plane OAB is perpendicular to the plane DFE

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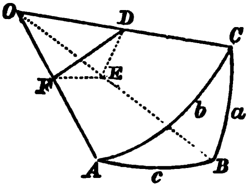
[*] This theorem can be easily remembered by its resemblance to (1) Art. 30.

(Ch. 17, VI.). The plane OBC is perpendicular to the plane OAB by hypothesis.

Thus two planes, DFE, and OBC, are perpendicular to a third plane OAB; therefore their line of intersection DE is also perpendicular to the plane OAB (Ch. 20, VI.).

Consequently DEF, and DEO, are right-angled triangles (Ch. Art. 6, VI.).



DFO and EFO are right-angled triangles by construction.

DFE is the *plane angle* of the diedral angle whose edge is OA, and therefore is equal to the angle A of the spherical triangle ABC (Ch. Arts. 39 and 45, VI. Prop. 16, VIII.).

Denote the sides opposite the angles of the spherical triangle by small letters of the same name.

$$\sin. a = \sin. BOC \ ((a) \text{ Art. } 103) = \frac{DE}{DO} \ ((1) \text{ Art. } 4)$$

$$= \frac{DF}{DO} \times \frac{DE}{DF};$$

but $\frac{DF}{DO} = \sin. COA = \sin. b$;

and $\frac{DE}{DF} = \sin. DFE = \sin. A$;

$\therefore$ (1) $\sin. a = \sin. b \sin. A$.

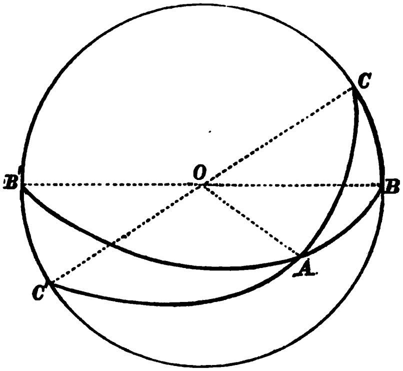
By drawing FD, and ED perpendicular to OC, it can be proved that

(2) $\sin. c = \sin. b \sin. C$.

110. In the preceding article the demonstration has been applied to a figure drawn to represent a triangle

of which the sides are each less than 90°, but the theorem is true for all right-angled triangles.

Let ABC represent a triangle on the surface of a



hemisphere. Let the three sides, a, b, and c, be each less than 90°. Then the angles, A and C, are each less than 90° (Art. 108). Complete the figure as in Art. 106.

Not considering the right angle, of the five parts (three sides and two angles) of each of the triangles BAC', $B'AC'$, and $B'AC$, *one* part, as in the triangle $B'AC$, is the same as a part of ABC; or *two* parts, as in the triangle $B'AC'$, are equal to parts in the triangle ABC, and the other parts are the supplements of parts of the triangle ABC. Therefore the theorem is true for these triangles (Art. 46, (*a*) Art. 103).

Thus in the triangle $B'AC$,

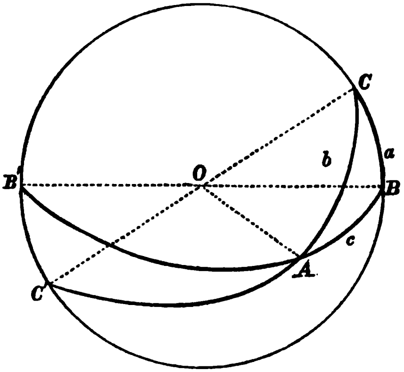
sin. $B'C =$ sin. a; sin. $AC =$ sin. b; sin. CAB' $=$ sin. A; but, according to previous article,

$$\text{sin. } a = \text{sin. } b \text{ sin. } A; \text{ substituting,}$$

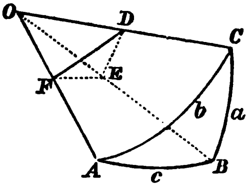
$$\text{sin. } B'C = \text{sin. } AC \times \text{sin. } CAB'.$$

In a similar manner the theorem may be proved true for the triangles BAC' and $B'AC'$.

If A and B were both right angles the theorem would still be true, for in that case AC and BC would be quadrants (Ch. Art. 89, VIII.), and sin. a, sin. b, and sin. A would each be 1; also c would equal C (Ch. 16, VIII.).



111. *In a right-angled spherical triangle, the cosine of the hypotenuse is equal to the product of the cosines of the other two sides.*



Let ABC be a triangle right-angled at B, and on the surface of a sphere whose centre is O, the vertex of the right triedral angle $OABC$. Then we are to prove that cos. $b =$ cos. a cos. c.

Construct the triangle DEF as in the figure of Art. 109.

$$\cos. b = \cos. AOC = \frac{OF}{OD} \text{ ((4) Art. 4)}$$

$$= \frac{OE}{OD} \times \frac{OF}{OE};$$

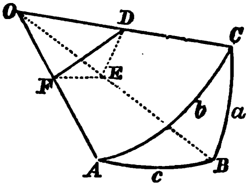
but $\frac{OE}{OD} = \cos. BOC = \cos. a$; and $\frac{OF}{OE} = \cos. AOB$

$$= \cos. c;$$

therefore $\cos. b = \cos. a \cos. c$.

This proposition may be proved to be true for all right-angled triangles by applying the principle of Art. 46, with regard to the cosine, to the figure of the preceding article.

112. *In a right-angled spherical triangle the tangent of either side about the right angle is equal to the product of the tangent of the opposite angle by the sine of the other side.*[*]



Let ABC be a spherical triangle, right-angled at B, formed by a right triedral angle whose vertex is at the centre, O, of the sphere.

Then we are to prove $\tan. a = \tan. A \sin. c$.

Construct the triangle DEF as in Art. 109.

Then DFO, and EFO, are triangles right-angled at F; and DEF, and DEO, are triangles right-angled at E. (See Art. 109.)

$$\tan. a = \tan. BOC = \frac{DE}{EO} \text{ ((2) Art. 4)};$$

$$\frac{DE}{EO} = \frac{DE}{EF} \times \frac{EF}{EO} = \tan. DFE \times \sin. AOB$$

$$= \tan. A \sin. c; \text{ therefore}$$

---

[*] This theorem may be remembered by its resemblance to (2) Art. 32.

(1) $\tan. a = \tan. A \sin. c.$

By drawing FD and ED at right angles to OC it may be proved

(2) $\tan. c = \tan. C \sin. a.$

113. *In a right-angled spherical triangle, the tangent of either side about the right angle is equal to the product of the cosine of the adjacent oblique angle by the tangent of the hypotenuse.**

Construct the figure as in Art. 109. Then we are to prove $\tan c = \cos. A \tan. b.$

$$\tan. c = \tan. AOB = \frac{EF}{FO} = \frac{EF}{FD} \times \frac{FD}{FO};$$

but $\dfrac{EF}{FD} = \cos. DFE = \cos. A;$

and $\dfrac{FD}{FO} = \tan. AOC = \tan. b;$

$\therefore$ (1) $\tan. c = \cos. A \tan. b.$

In a similar manner it may be proved

(2) $\tan. a = \cos. C \tan. b.$

The theorem of this article and the theorem of the preceding article can both be shown to be true for all right-angled triangles, by applying Art. 46 to the figure of Art. 110.

114. The *Circular Parts* of a right-angled spherical triangle are the *sides about the right angle*, the *complements* of the *hypotenuse* and the *oblique angles.*

The right angle is *not* considered one of the circular parts.

Of the five circular parts any one may be taken as the *middle part*, and then the *two next* to this, one on the one hand and the other on the other (not counting the right angle as a part) are called *adjacent parts*, and the remaining two, each separated from the middle part by an adjacent part, are called *opposite parts*.

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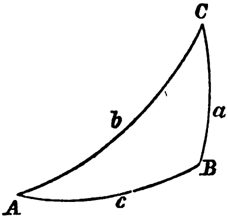
* **This theorem may be remembered by its resemblance to (2) Art. 29.**



Thus a, c, $90^\circ - b$, $90^\circ - A$, and $90^\circ - C$, are the *circular parts* of the triangle ABC right-angled at B.

If a is the *middle* part,
adjacent parts are c, $90^\circ - C$; *opposite* parts are $90^\circ - A$, $90^\circ - b$.

If c is the *middle* part,
adjacent parts are a, $90^\circ - A$; *opposite* parts are $90^\circ - b$, $90^\circ - C$.

If $90^\circ - b$ is the *middle* part,
adjacent parts are $90^\circ - A$, $90^\circ - C$; *opposite* parts are a, c.

If $90^\circ - A$ is the *middle* part,
adjacent parts are $90^\circ - b$, c; *opposite* parts are a, $90^\circ - C$.

If $90^\circ - C$ is the *middle* part,
adjacent parts are a, $90^\circ - b$; *opposite* parts are c, $90^\circ - A$.

115. *Napier's rule of the Circular Parts.*

The sine of the middle part is equal to the product of the tangents of the adjacent parts; and *the sine of the middle part is equal to the product of the cosines of the opposite parts.*

Let ABC be a spherical triangle right-angled at B.

We shall take each circular part in succession as a middle part; beginning with $90^\circ - b$, and next taking $90^\circ - C$, and so on.

Taking $90^\circ - b$ as the middle part we are to prove $\sin.(90^\circ - b) = \tan.(90^\circ - A)\tan.(90^\circ - C)$; or (Art. 5),

(1) $\cos. b = \cot. A \cot. C$.

Now, according to Art. 112,

(a) $\tan. a = \tan. A \sin. c$, and

(b) $\tan. c = \tan. C \sin. a$.

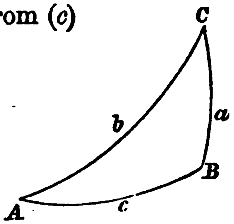
Divide equation (*b*) by equation (*a*), putting the second member of (*b*) over the first member of (*a*).

$$\frac{\tan. C \sin. a}{\tan. a} = \frac{\tan. c}{\tan. A \sin. c},$$

$$\tan. C \cos. a = \frac{1}{\tan. A \cos. c} \text{ ((}b\text{) Art. 64), or}$$

$$(c) \cos. a \cos. c = \frac{1}{\tan. A \tan. C} = \cot. A \cot. C$$

((*f*) Art. 64); but, according to Art. 111,
$\cos. b = \cos. a \cos. c$; therefore from (*c*)
$\cos. b = \cot. A \cot. C$. Again,
$\sin. (90° - b) = \cos. a \cos. c$, or
(2) $\cos. b = \cos. a \cos. c$ (proved under Art. 111).



Next take $90° - C$ *as middle part*, and we are to prove
$\sin. (90° - C) = \tan. a \tan. (90° - b)$, or
(3) $\cos. C = \tan. a \cot. b$, and
$\sin. (90° - C) = \cos. c \cos. (90° - A)$, or
(4) $\cos. C = \cos. c \sin. A$.

To prove (3) we have from Art. 113
$\cos. C \tan. b = \tan. a$; therefore

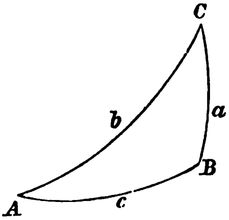
$$\cos. C = \frac{\tan. a}{\tan. b} = \tan. a \cot. b \text{ ((}f\text{) Art. 64).}$$

To prove (4)

$$(d) \cos. C = \tan. a \cot. b = \frac{\sin. a}{\cos. a} \times \frac{\cos. b}{\sin. b}$$

((*b*), (*d*), Art. 64);

$$\sin. A = \frac{\sin. a}{\sin. b} \text{ (Art. 109).}$$



Also cos. $b =$ cos. a cos. c (2); substituting these values for their equivalents in second member of (d)

$$\cos. C = \cos. c \sin. A.$$

Let a be the *middle part*, then we are to prove

(5) sin. $a =$ tan. c tan. $(90° - C) =$ tan. c cot. C;

(6) sin. $a =$ cos. $(90° - b)$ cos. $(90° - A)$
$= \sin. b \sin. A.$

To prove (5)

$$\sin. a = \frac{\tan. c}{\tan. C} \text{ (Art. 112)}$$
$$= \tan. c \cot. C;$$

(6) is proved under Art. 109.

Let c be the *middle part*, then

(7) sin. $c =$ tan. a cot. A; proved in the same manner as (5).

(8) sin. $c =$ sin. b sin. C (Art. 109).

Lastly, let $90° - A$ be the *middle part*, then

(9) cos. $A =$ cot. b tan. c; proved by Art. 113; see proof of (3).

(10) cos. $A =$ cos. a sin. C;

(e) $\cos. A = \dfrac{\cos. b}{\sin. b} \times \dfrac{\sin. c}{\cos. c}$ ((9), and (b) and (d) of Art. 64),

$\sin. C = \dfrac{\sin. c}{\sin. b}$; also cos. $b =$ cos. a cos. c; substituting these values in (e)

$$\cos. A = \cos. a \sin. C.$$

CHAPTER XII.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

Art. 116. Any *two* parts (in addition to the right angle) of a right-angled spherical triangle being known, the triangle can be solved, and the *unknown* parts can be obtained in terms of the *known* parts, by applying Napier's rule.

117. To find any unknown part from two known parts, care should be taken in applying Napier's rule to make such a selection of one of the three parts for a middle part that the other two should be either both adjacent parts or both opposite parts.

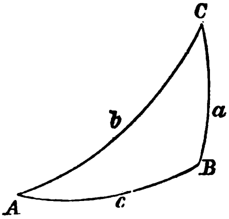
118. Care should also be taken to observe the *signs* of the trigonometric functions, as the sign of the result will determine generally whether the arc or angle is $< 90°$ or $> 90°$ (Art. 46).

119. The two parts, which may be given (in addition to the right angle) to solve a right-angled spherical triangle may be: 1. the *hypotenuse* and a *side;* 2. the *hypotenuse* and *an angle;* 3. the *two sides about the right angle;* 4. a *side* and an *adjacent angle;* 5. a *side* about the right angle and the *opposite angle;* 6. the *two angles.*

120. Case I.—*The hypotenuse and a side,* of a

right-angled spherical triangle, *being known, to solve the triangle.*

In the triangle ABC, right-angled at B, suppose b and a to be known. It is required to find c and the angles A and C in terms of a and b.



To find c take $90^\circ - b$ as the *middle* part, and a and c as the *opposite* parts; then

$$\cos. b = \cos. a \cos. c \text{ (Art. 115)};$$

$$(a) \text{ or } \cos. c = \frac{\cos. b}{\cos. a}.$$

To find A take a as a *middle* part, $90^\circ - A$ and $90^\circ - b$ as *opposite* parts; then

$$\sin. a = \sin. A \sin. b; \text{ whence}$$

$$(b) \ \sin. A = \frac{\sin. a}{\sin. b}.$$

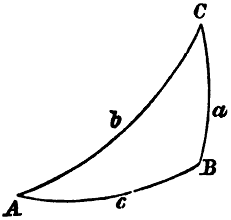
To find C take $90^\circ - C$ as a *middle* part, a and $90^\circ - b$ as *adjacent* parts; then

$$(c) \ \cos. C = \tan. a \cot. b.$$

As from equation (a) c is obtained by means of its cosine, the sign of $\frac{\cos. b}{\cos. a}$ will determine whether c is $< 90^\circ$ or $> 90^\circ$ (Art. 46). If a and b are both $< 90^\circ$ or both $> 90^\circ$, $\frac{\cos. b}{\cos. a}$ will be positive, and cos. c will also be positive, and therefore c will be $< 90^\circ$. If a and b are *not* of the *same species* (Art. 107), $\frac{\cos. b}{\cos. a}$ will be negative, and consequently cos. c will be negative, and, therefore, c will be $> 90^\circ$.

Though from (b) A is obtained by means of its sine, A can have only one value, since A and a are of the same species (Art. 108).

Also as C is found from its cosine, the sign of tan. a cot. b will determine whether C is $< 90°$ or $> 90°$. Again C and c must be of the same species (Art. 108).



As check on work, form an equation in which only the three required parts occur. In the present case after c, A, and C are found, if the results are correct, the equation made by applying Napier's rule to $90° - C$ as a middle part, c and $90° - A$ as opposite parts, that is

$$\cos. C = \cos. c \sin. A,$$

should be a true equation.

If it prove not to be a true equation, there must be some error in the previous calculations.

121. CASE II.—The *hypotenuse and an angle* of a right-angled spherical triangle *being known*, to *solve the triangle.*

B being the right angle, suppose b and A are given to find C, a, and c.

Take $90° - b$ as a *middle* part, $90° - A$ and $90° - C$ as *adjacent* parts; then

$$\cos. b = \cot. A \cot. C \text{ (Art. 115)};$$

$$(a)\ \cot. C = \frac{\cos. b}{\cot. A}.$$

To find a, use it as a *middle* part, and $90° - A$, $90° - b$ as *opposite* parts.

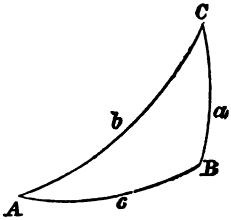
$$(b)\ \sin. a = \sin. b \sin. A.$$

To find c, take $90°-A$ as *middle* part, and c, $90°-b$ as *opposite* parts.

$$\cos. A = \tan. c \cot. b, \text{ whence}$$

$$(c)\ \tan. c = \frac{\cos. A}{\cot. b}.$$

As C is found by means of its cotangent, and as c is found by means of its tangent, the sign of the equivalent fraction in each case will determine whether the quantity is $< 90°$ or $> 90°$ (Art. 46).



Though a is found by means of its sine, it can have but one value, being $< 90°$ or $> 90°$, according as A is $< 90°$ or $> 90°$ (Art. 108).

As check on work, use the equation

$$\sin. a = \cot. C \tan. c.$$

122. CASE III.—*The two sides about the right angle* of a right-angled spherical triangle *being known, to solve the triangle.*

Suppose a and c, about the right angle B, to be known, it is required to find A, b, and C.

To find A take c as a *middle* part, and $90°-A$ and a as *adjacent* parts; then,

$$\sin. c = \cot. A \tan. a \text{ (Art. 115)};$$

$$(a)\ \cot. A = \frac{\sin. c}{\tan. a}.$$

To find b, take it as middle part,

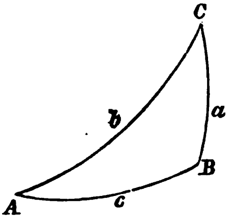
(b) $\cos. b = \cos. a \cos. c.$

To find C take a as a *middle* part, and c, $90°-C$ as *adjacent* parts.

$\sin. a = \cot. C \tan. c$, or

(c) $\cot. C = \dfrac{\sin. a}{\tan. c}$.

As A and C are each found by means of its cotangent, and as b is found by means of its cosine, each of these quantities can have only one value, which will be determined by the sign of its trigonometric function (Art. 46).



As check on work, use the equation

$$\cos. b = \cot. A \cot. C.$$

123. CASE IV.—*A side about the right angle*, of a right-angled spherical triangle, *and the adjacent oblique angle being known*, to *solve the triangle.*

B being the right angle, suppose c and A are known; it is required to find b, C, and a.

To find b, take $90°-A$ as the *middle* part, and $90°-b$, c, as *adjacent* parts.

$\cos. A = \cot. b \tan. c$ (Art. 115), or

(a) $\cot. b = \dfrac{\cos. A}{\tan. c}$.

To find C, take $90°-C$ as *middle* part, and c, $90°-A$ as *opposite* parts.

(b) $\cos. C = \cos. c \sin. A$.

To find a, take c as *middle* part, and a, $90°-A$ as *adjacent* parts.

$\sin. c = \tan. a \cot. A$, or

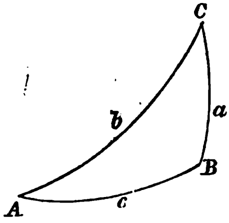
(c) $\tan. a = \dfrac{\sin. c}{\cot. A}$.

As the required parts are found by means of the cotangent, cosine, and tangent respectively, each can have but one value, which will be determined by the sign of its trigonometric function (Art. 46).

As check on work, use the equation

$\cos. C = \cot. b \tan. a.$

124. CASE V.—*A side about the right angle, of a right-angled spherical triangle, and the opposite angle being known, to solve the triangle.*



B being the right angle, a and A are given; it is required to find b, C, and c.

To find b, take a as a *middle* part, and $90° - A$, $90° - b$ as *opposite* parts.

$$\sin. a = \sin. b \sin. A \text{ (Art. 115)};$$

$$(a)\ \sin. b = \frac{\sin. a}{\sin. A}.$$

To find C, take $90° - A$ as *middle* part, and a, $90° - C$ as *opposite* parts.

$$\cos. A = \cos. a \sin. C; \text{ whence}$$

$$(b)\ \sin. C = \frac{\cos. A}{\cos. a}.$$

To find c, make it the *middle* part, and a, $90° - A$ *adjacent* parts.

$$(c)\ \sin. c = \tan. a \cdot \cot. A.$$

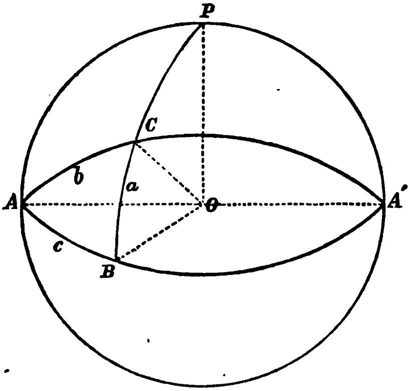
As, in this case, each of the required parts is obtained by means of its sine, each part will have two values (Art. 46), and there will be solutions answering to two triangles, each of which will contain the given parts.

The required parts of one will be supplements of the required parts of the other.

As check on work, use the equation

$\sin. c = \sin. b \sin. C.$

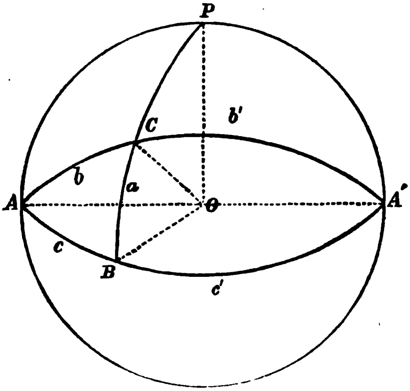
125. That there will be two right-angled spherical triangles containing the same given parts, when those parts are a side and the opposite angle, will be evident from the accompanying figure.



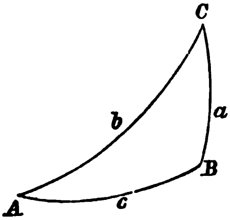
Let $BA'OAC$ be an ungula in a hemisphere of which the base is the circle, APA', and the centre O. The lune $ABA'C$ will be the base of the wedge (Ch. Art. 90, VIII.).

Let a plane $BCPO$ be passed through the pole, P, of the circle ABA', through the centre, O, and any point, B, of the arc ABA' (Ch. Art. 33, VIII.). The wedge will be divided into two right triedral angles, $OBAC$, $OBA'C$, having the face angle, BOC, in common, and having the diedral angles, AO and OA', the same, since these are parts of the diedral angle AA'. The remaining parts of the one are the supplements of the corresponding parts of the other.

The lune will be divided into two right-angled triangles, corresponding to the right triedral angles, having the side a in common, and having the angle A equal to the angle A' (Ch. 16, VIII.), and having the remaining parts of the one the supplements of the corresponding parts of the other.



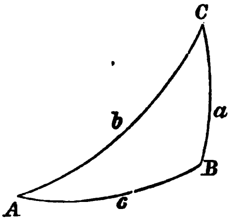
This case of the solution of right-angled spherical triangles is called the *ambiguous* case.



126. Case VI.—*The two angles of a right-angled spherical triangle being known, to solve the triangle.*

B being the right-angle, suppose A and C to be known; it is required to find b, a, and c.

To find b, make $90°-b$ the *middle* part; then



(*a*) $\cos. b = \cot. A \cot. C$ (Art. 115).

To find a, make $90°-A$ the *middle* part.

$$\cos. A = \sin. C \cos. a, \text{ or}$$

$$(b) \quad \cos. a = \frac{\cos. A}{\sin. C}.$$

To find c, make $90°-C$ the *middle* part.

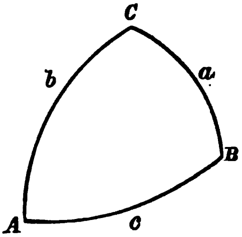
$$\cos. C = \cos. c \sin. A, \text{ or}$$

$$(c) \quad \cos. c = \frac{\cos. C}{\sin. A}.$$

As check on work, use the equation

$$\cos. b = \cos. a \cos. c.$$

Suppose A, B, C to be a right-angled spherical triangle of which the hypotenuse, b, is 100° and the side, a, is 60°. Required to find the other parts.



$$\cos. b = \cos. a \cos. c.$$

$$\therefore \cos. c = \frac{\cos. b}{\cos. a}.$$

$$= \frac{\cos. 100°}{\cos. 60°} = - \text{ quantity.}$$

$$\therefore c > 90°.$$

$$\text{Log. cos. } 100° = 9.239670$$

$$\text{Log. cos. } 60° = 9.698970$$

$$\text{Log. cos. } 69° \, 40' \, 40.7'' \qquad 9.540700$$

$$\therefore c = 110° \, 19' \, 19.3''$$

$$\sin. a = \sin. b \sin A.$$

$$\therefore \sin. A = \frac{\sin. a}{\sin. b} = \frac{\sin. 60°}{\sin. 100°}$$

A of same species as a.

$$\therefore A < 90°.$$

$$\text{Log. sin. } 60° = 9.937531$$

$$\text{Log. sin. } 100° = 9.993351$$

$$\text{Log. sin. } 61° \, 34' \, 6\tfrac{2}{3}'' \qquad 9.944180$$

$$\cos. C = \tan. a \cot. b.$$
$$= \tan. 60^\circ \cot. 100^\circ = -\text{quantity}.$$

$\therefore C > 90^\circ.$

$$\begin{array}{ll} \text{Log. tan. } 60^\circ = & 10.238561 \\ \text{Log. cot. } 100^\circ = & \underline{9.246319} \\ \text{Log. cos. } 72^\circ\ 13'\ 2.27'' & 9.484880 \end{array}$$

$\therefore C = 107^\circ\ 46'\ 57.73''.$

(*Check.*)

$$\cos. C = \cos. c \sin. A.$$

$$\begin{array}{rl} \text{Log. cos. } c = & 9.540700 \\ \text{Log. cos. } A = & \underline{9.944180} \\ & 9.484880 \\ \text{Log. cos. } C = & 9.484880 \end{array}$$

(Examples worked by six-place logarithm tables, and answers given to nearest tenth of second.)

Solve a right-angled triangle when there are given

Hypotenuse and a side.

EXAMPLE 1. Hypotenuse, 72°; side, 13°. *Ans.* Side, 71° 30′ 35″; angles, 13° 40′ 54″; 85° 41′ 53″.

2. Hypotenuse, 64° 40′; side, 137° 50′.

Hypotenuse and an angle.

3. Hypotenuse, 73°; angle, 15°. *Ans.* Angle, 85° 31′ 14″; sides, 14° 19′ 48.7″; 72° 26′ 11.8″.

4. Hypotenuse, 92°; angle, 95°. *Ans.* Angle, 68° 15′ 10″; sides, 95° 23′ 5″; 68° 9′ 55.6″.

Two sides about the right angle.

5. 63° and 42°. *Ans.* Hypotenuse, 70° 16′ 57⅓″; angles, 71° 10′ 25¼″; 45° 18′ 2″.

6. 16° and 116°. *Ans.* Hypotenuse, 114° 55′ 20.5″; angles, 17° 41′ 40″; 97° 39′ 24.4″.

7. 100° and 135°.

A side about the right angle and an adjacent angle.

8. Side, 39°; angle, 61°. *Ans.* Hypotenuse, 59° 5′ 29.4″; side, 48° 37′ 34.3″; angle, 47° 10′ 45.5″.

9. Side, 100° 35′; angle, 50° 2′. *Ans.* Hypotenuse, 96° 50′ 37.6″;
side, 49° 32′ 55.3″; angle, 98° 5′ 31.5″.

10. Side, 112° 4′; angle, 100° 6′.

A side about the right angle and an opposite angle.

11. Side, 25° 16′; angle, 36° 13′.
Ans. Hypotenuse, 46° 15′ 15.5″; 133° 44′ 44.5″;
side, 40° 7′ 40″; 139° 52′ 20″; angle, 63° 8′ 36.4″. 116° 51′ 26.6″.

12. Side, 114° 2′; angle, 102° 15′.
Ans. Hypotenuse, 69° 9′ 42.5″; 110° 50′ 17.5″;
side, 29° 8′ 13.2″; 150° 51′ 46.8″; angle, 31° 28′ 52.6″. 148° 36′ 7.4″.

13. Side, 62° 10′; angle, 74° 1′.

Two angles.

14. 56° and 40°. *Ans.* Hypotenuse, 36° 30′ 3¼″;
sides, 29° 32′ 49.2″; 22° 28′ 45.6″.

15. 131° 20′ and 110° 15′. *Ans.* Hypotenuse, 71° 8′ 57.6″;
sides, 134° 44′ 41″; 117° 26′ 53.9″.

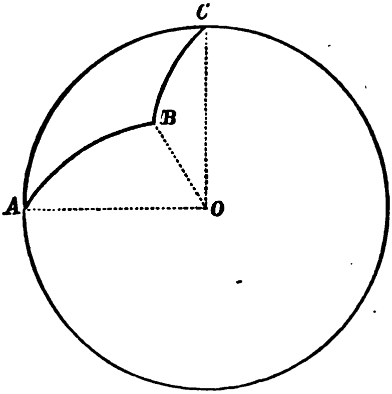
16. 50° and 120°.

CHAPTER XIII.

QUADRANTAL TRIANGLES.

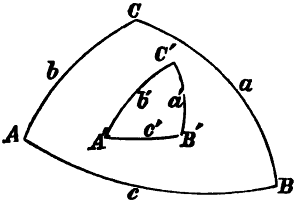
ART. 127. A quadrantal triangle is formed by a triedral angle (at the centre of a sphere), one of whose face angles is a right angle (Art. 105).

Thus suppose $OABC$ is a triedral angle (at the centre, O, of a sphere), having one of its face angles, AOC, a right angle. Then the planes of the faces of the triedral angle will form, by their intersection with the surface of the sphere, a quadrantal triangle, ABC, of which the side, AC, which measures the right angle, AOC, is a quadrant (Ch. Art. 53, II.).



128. *The polar triangle of a quadrantal triangle is a right-angled triangle.*

Let ABC be a quadrantal triangle, having its side, AC or b, a quadrant. Let $A'B'C'$ be the *polar* triangle of ABC (Ch. Art. 67, VIII.). Denote the sides of the triangle ABC by a, b, c, and the sides of $A'B'C'$ by a', b', and c', respectively.



Then $A'B'C'$ is a right-angled triangle, right-angled at B'.

$B' + b = 180°$ (Ch. Arts. 69 and 70, VIII.); therefore

$$B' = 180° - b = 180° - 90° = 90°.$$

129. *Any two parts of a quadrantal triangle, in addition to the side which is a quadrant, being known, we can solve the polar right-angled triangle.*

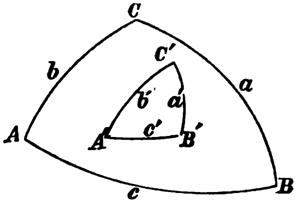
For the two given parts of the quadrantal triangle are the supplements of two parts of the polar triangle (Ch. 18, VIII.). Two parts of the polar triangle, in addition to the right angle (Art. 128), will therefore be known, and the polar triangle can be solved (Art. 116).

Thus let ABC be a quadrantal triangle, having the side, b, a quadrant. Let $A'B'C'$ be the polar triangle of ABC. Denote the sides of the triangle ABC by a, b, c, and the sides of $A'B'C'$ by a', b', and c', respectively.

(1) $a' = 180° - A$; (2) $b' = 180° - B$; (3) $c' = 180° - C$; also (4) $A' = 180° - a$; (5) $B' = 180° - b = 90°$; (6) $C' = 180° - c$ (Ch. Art. 70, VIII.).

From equations (1) to (6) inclusive, it is evident that if we know any two parts of ABC, besides b, we know two parts of $A'B'C'$ besides the right angle B'.

130. The polar right-angled triangle being solved, and all its parts being known, all the parts of the quadrantal triangle will be known, as the sides of the one are the supplements of the angles of the other (Ch. 18, VIII.).



131. To solve a *quadrantal* triangle we derive from Arts. 128, 129, and 130, the following rule:

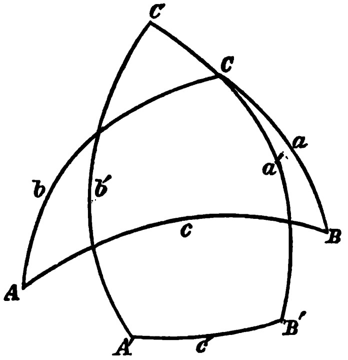
Take the supplements of the given parts of the quadrantal triangle for the given parts of the polar right-angled triangle; solve the polar triangle, and take the supplements of the parts found as the required parts of the quadrantal triangle.

If in the *quadrantal triangle* we have given, in addition to the quadrant, the other two *sides*, in the *polar right-angled triangle* we have given, besides the right angle, the two *angles* ((4) and (6), Art. 129).

If in the *quadrantal triangle* we have given any *two angles*, in the *polar right-angled triangle* we have given *two* corresponding *sides* ((1), (2) and (3), Art. 129).

If in the *quadrantal triangle* we have given, in addition to the quadrant, a side and an angle, in the *polar right-angled triangle* we have given, besides the right angle, an angle and a side ((4) and (1) or (4) and (2), etc., Art. 129).

In any example all these relations will be best seen by drawing the quadrantal triangle and its polar triangle.



Example. Solve a quadrantal triangle of which there are given, in addition to the side which is a quadrant, a side equal to 64°, and an angle between the side and quadrant equal to 120°.

Let $b = 90°$; $a = 64°$;
$C = 120°$.
$\therefore B' = 90°$; $A' = 116°$;
$c' = 60°$.

Solution of $A'B'C'$ falls under Case IV. (Art. 123).

$\cos. A' = \cot. b' \tan. c'$.

$$\therefore \cot. b' = \frac{\cos. A'}{\tan. c'} = \frac{\cos. 116°}{\tan. 60°} = -\text{quantity} \therefore b' > 90°.$$

$$\begin{aligned} \text{Log. cos. } 116° &= 9.641842 \\ \underline{\text{Log. tan. } 60°} &= \underline{10.238561} \\ \text{Log. cot. } 75° \ 47' \ 49.43'' &= 9.403281 \end{aligned}$$

$b' = 104° \ 12' \ 10.57''$.

$\cos. C' = \sin. A' \cos. c'$

$= \sin. 116° \cos. 60° = +$ quantity.

$\therefore C' < 90°$.

$$\begin{aligned} \text{Log. sin. } 116° &= 9.953660 \\ \underline{\text{Log. cos. } 60°} &= \underline{9.698970} \\ \text{Log. cos. } 63° \ 17' \ 41.95'' &= 9.652630 \end{aligned}$$

$C' = 63° \ 17' \ 41.95''$.

$\sin. c' = \cot. A' \tan. a'$

$$\therefore \tan. a' = \frac{\sin. c'}{\cot. A} = \frac{\sin. 60°}{\cot. 116°}$$

$= -$ quantity.

$\therefore a' > 90°$.

$$\begin{aligned} \text{Log. sin. } 60° &= 9.937531 \\ \underline{\text{Log. cot. } 116°} &= \underline{9.688182} \\ \text{Log. tan. } 60° \ 36' \ 44.8'' &= 10.249349 \end{aligned}$$

$a' = 119° \ 23' \ 15.2''$.

(*Check.*) $\cos. C' = \cot. b' \tan. a'$.

$$\begin{aligned} \text{Log. cot. } b' &= 9.403281 \\ \underline{\text{Log. tan. } a'} &= \underline{10.249349} \\ \text{Log. cos. } C' &= 9.652630 \end{aligned}$$

$b' = 104° \ 12' \ 10.57''$ $\therefore B = 75° \ 47' \ 49.43''$.

$C' = 63° \ 17' \ 41.95''$ $\therefore c = 116° \ 42' \ 18.05.''$

$a' = 119° \ 23' \ 15.2''$ $\therefore A = 60° \ 36' \ 44.8''$.

Solve a quadrantal triangle when there are given, in addition to the side which is a quadrant:

EXAMPLE 1. The two angles adjacent to the quadrant, 75° and 104°.

Ans. Angle = 86° 24′ 36.5″; sides = 75° 25′ 34.8″ and 103° 32′ 27″.

2. The two other sides, 70° and 105°.

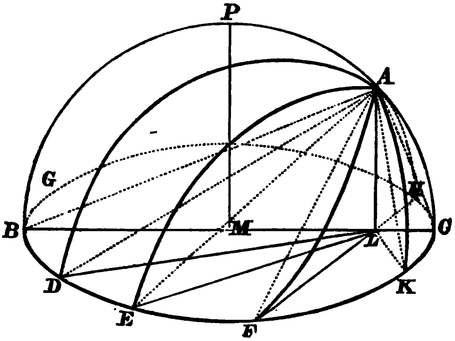
Ans. Angles, 84° 24′ 11.9″; 69° 15′ 45.4″; 105° 59′ 15.3″.

3. A side 50°, angle between the side and quadrant 125°.

CHAPTER XIV.

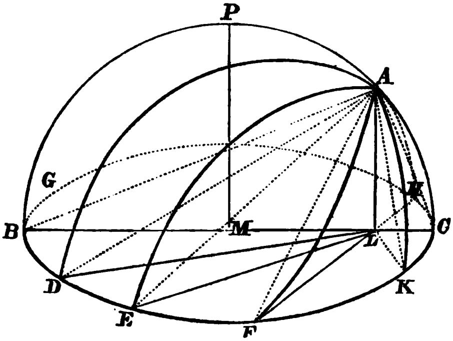
THEOREMS OF OBLIQUE-ANGLED TRIANGLES.

ART. 132. *If from a point in the surface of a hemisphere, which is not the pole of its base, arcs of great circles are drawn to the circumference of the great circle, which is its base, of all these arcs the* GREATEST *is that which passes* THROUGH *the* POLE, *the* LEAST *is that which when produced passes through the pole, and of the others that which is* NEARER *to the greatest is* GREATER *than the* MORE REMOTE*; and from this point* EQUAL ARCS *of great circles can be drawn only* IN PAIRS.



Thus, let the figure *PBECG* represent a hemisphere having for its base the great circle *BECG*.

Let *P* be the *pole* of the base (Ch. Art. 27, VIII.). Suppose *A* to be any other point on the surface of the hemisphere. From *A* let the arcs *APB*, *AC*, *AD*, *AE*, *AF*, and *AK*, be drawn to the circumference, *BECG*.



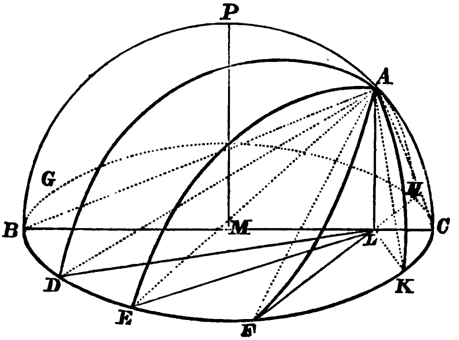
Then, of these arcs *APB* is the greatest, *AC* is the least, and *AD*, which is nearer to *APB*, is greater than *AE*, which is more remote; *AE* than *AF*, etc.

Let *M* be the centre of the sphere of which the figure represents the hemisphere. Join *PM*; then *PM* is perpendicular to the plane of the circle, *BECG* (Ch. Art. 27, VIII.), and the planes, *BPCM* and *BFCM*, are perpendicular to each other (Ch. 17, VI.); also the arcs *BAC* and *BFC* are perpendicular to each other (Ch. Art. 58, VIII.). *BC* is the diameter of the sphere common to the two semicircles *BPCM* and *BFCM* (Ch. Art. 32, VIII.).

Now, let a perpendicular be let fall from *A* upon the plane of the circle *BECG*. It will lie in the plane *BACM* (Ch. Art. 51, VI.), and at *L* will meet at right angles the line of intersection, *BC*, of the two planes *BACM* and *BFCM* (Ch. Art. 6, VI.).

Draw the straight lines *LD*, *LE*, *LF*, and *LK*, in the plane *BFCG*; also draw the straight lines *AB*, *AD*, *AE*, *AF*, *AK*, and *AC*.

Now, *M* is the centre of the circle *BFCG* (Ch. Art. 26, VIII.), and *L* is a point in the diameter which is not the centre; therefore *LB* is the greatest line which can be drawn from *L* to the circumference *BFCG*, *LC* is the least, and *LD*, which is nearer to *LB*, is greater than *LE*, *LE* than *LF*, etc. (Euc. 7, III. Ch. 17 and 24, I.).

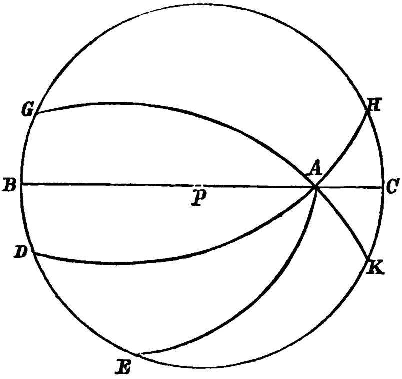


Therefore of the straight lines, drawn from the point *A* to the points *B*, *D*, *E*, *F*, *K*, and *C*, *AB* is the greatest, *AC* is the least, and *AD* is greater than *AE*, *AE* than *AF* (Euc. 47, I. Ch. 4, VI.).

Consequently, since in equal circles, or in the same circle, greater chords cut off greater arcs, the arcs being less than semicircumferences (Ch. 6, II.) of all the great circle arcs, drawn from *A* to points on the circumference *BFCG*, *APB* is the greatest, *AC* is the least, *AD* is greater than *AE*, *AE* than *AF*, and *AF* than *AK*, these arcs being arcs of equal circles and all less than semicircumferences (Ch. Art. 29 and 32, VIII.).

Again, from the point *L* equal straight lines can be drawn to the circumference *BFCG* only in pairs (Euc. 7, III. Ch. 20, I.). Consequently, equal straight lines, and equal arcs of great circles, can be drawn from *A* to the circumference, *BFCG*, only in pairs (Euc. 47, I. and 28, III. Ch. 4, VI. and 5, II.).

133. Let every point on the surface of the hemisphere be projected upon the plane of its base (Ch. Art. 56, VI.); that is, let the hemisphere itself be projected upon the plane of its base.

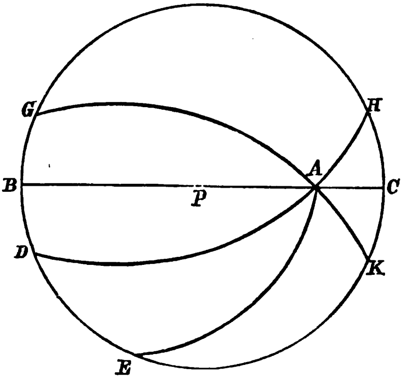


Let the figure *BECG* represent the hemisphere thus projected, with the arcs also projected in the lines *BPC*, *DAH*, *AK*, and *AE*.

The arc *BPC* of the preceding figure will be projected in the straight line *BC*. Therefore the straight line *BPC* will represent an arc on the surface of the hemisphere perpendicular to the arc *BECG*.

By this method of projection *ACK*, *ACE*, *ACD*, and *ABD*, *ABE*, etc., are made to represent triangles right-angled at *C* and at *B*, respectively.

134. *If arcs* of great circles are drawn to the circumference of the base of a hemisphere, from a point on the *surface not the pole of the base*, of the two adjacent *angles made by* these arcs *with the circumference*



of the *base*, the *one on the side toward the longer perpendicular* is an *obtuse* angle, and *the other* angle on the side *toward the shorter perpendicular* is *acute*.

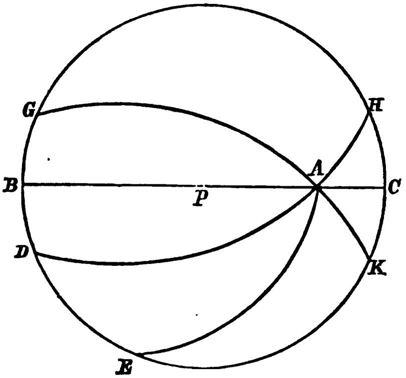
Let the hemisphere be represented as projected on the plane of its base, and let *AB*, *AD*, *AE*, *AK*, be arcs drawn from *A* on the surface of the hemisphere (not the pole of the base) to the circumference *BECG*. Of these arcs let *BAC* be the one passing through the pole of the base. *BAC* will be perpendicular to the arc *BECG* (Ch. Art. 58, VIII.).

In the right-angled triangle *ADB*, *AB* is $>$ *AD* (Art. 132);

$\therefore$ ADB is $>$ ABD (Ch. 26, VIII.), i. e., is $> 90°$, or is an obtuse angle.

Again, in the right-angled triangle ADC, AC is $<$ AD (Art. 132);

$\therefore$ the angle ADC is $<$ the angle ACD (Ch. 26, VIII.), i. e., $< 90°$, or is an acute angle.

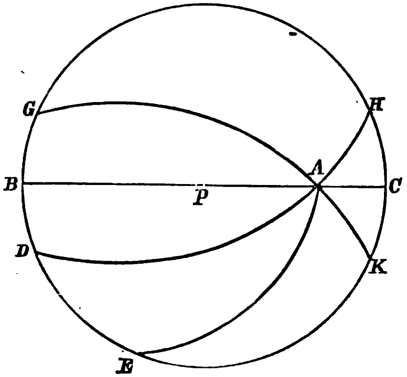


In a similar manner it may be proved that the angles AEB, AKB, are obtuse, and that the angles AEC and AKC are acute.

135. A *perpendicular upon an arc*, which coincides with a side of an oblique-angled spherical triangle, drawn *from an angle opposite* this side, falls *within* the triangle, if the *other two angles* are *both acute*, or are both *obtuse;* but falls *without* if *one* of these angles is *acute* while the *other* is *obtuse.*

Let HAK, HAE, etc., be triangles represented on the surface of a hemisphere which is projected on the plane of its base (Art. 133). Let these triangles have the common vertex A, not the pole of the base. Let

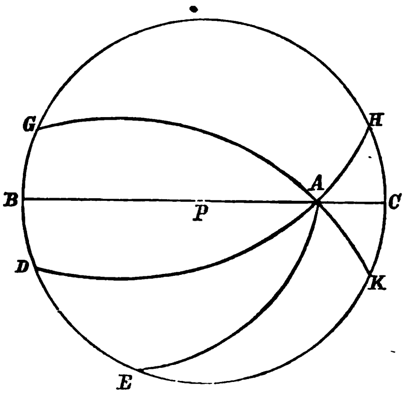
P be the pole of the base. Let the great-circle arc, *BAC*, be drawn through *P*, and let it meet the circumference of the base in two points, *B* and *C*. The angles at *B* and *C* will be right angles (Ch. Art. 58, VIII.).



Suppose the triangle *HAK* to have the angles *H* and *K* acute; also suppose the angles *AEK*, *ADK*, and *AGH* are acute.

As the angle *C* is a right angle, *ACH* is greater than *H*; therefore *AC* is less than *AH* (Ch. 26, VIII.), and must be the shorter perpendicular (Art. 132), for in the same manner it may be proved less than *AK* or *AE*, or less than *AD* or *AG*; therefore it is the least arc from *A*, and it is perpendicular by hypothesis. Now, as *AC* is the shorter perpendicular, the acute angle *H* lies toward *AC*; for the same reason, the acute angle *K* lies toward *AC* (Art. 134); that is, *C* lies between *H* and *K*, or the shorter perpendicular, *AC*, falls within the triangle *AKH*. In the same manner it may be shown to fall within the triangle *HAE*.

AC having been shown to be the shorter perpendicular, AB must be the longer perpendicular. As by hypothesis ADE is acute, ADB must be obtuse, and must lie toward the longer perpendicular (Art. 134); and, since AGH is acute, AGB must be obtuse, and must lie toward the longer perpendicular; therefore AB lies within the triangle ADG, of which the angles AGB and ADG are obtuse. In the same manner it may be proved that AB lies within the triangle AEG, the angles AGE and AEG being obtuse.



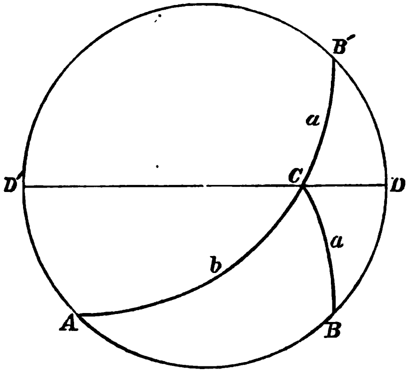
Next let AEK be a triangle having the angle AEK acute, but the angle AKE obtuse; then both perpendiculars, AC and AB, lie without the triangle.

Since AEK is acute, AC lies to the right of AE (Art. 134). Since AKE is obtuse, AKC must be acute, and AC lies to the right of AK; therefore AC, the shorter perpendicular, lies without the triangle.

Again, since AKE is obtuse, AB lies to the left of AK. Since AEK is acute, AEB must be obtuse, and

AB lies to the left of AE; therefore the longer perpendicular also lies without the triangle.

136. In a spherical triangle—1. the SUM of TWO ANGLES is 180°, if the SUM of the OPPOSITE SIDES is 180°; 2. is LESS than 180°, if the SUM of the OPPOSITE SIDES is LESS than 180°; 3. is greater than 180°, if the SUM of the OPPOSITE SIDES is GREATER than 180°.



Suppose ABC to be a spherical triangle projected on the plane of the base of the hemisphere $ABDB'$. Let $D'D$ be the projection of the arc drawn through C perpendicular to the arc AB produced.

1. Suppose $(a+b)=180°$; then $(A+B)=180°$.

If $a=b$, then a and b each $=90°$; and A and B each $=90°$ (Ch. Arts. 38 and 58, VIII.).

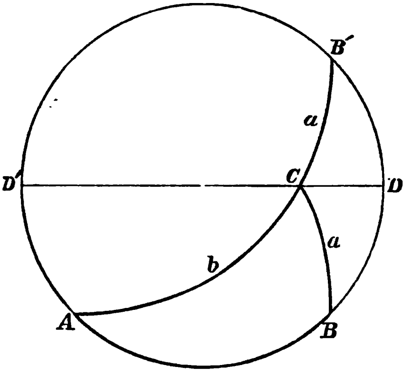
$\therefore A+B=180°$.

Let a and b be unequal.

Then a and b must lie on the same side of DD'.

For if b were on one side of DD' and a on the other side, occupying the positions CA and CB' respectively, they would together form an arc, ACB', equal to a semicircle, and the figure $ABB'C$ would be a *lune* and *not a triangle* (Ch. Arts. 32 and 90, VIII.).

Therefore a and b both lie on the same side of DD'. Produce AC to meet the circumference of the base of the hemisphere in B'. Then ACB' is a semicircumference, and is equal to 180° (Ch. Art. 32, VIII.).



Then $b + CB' = 180° = b + a$;

$\therefore CB' = a = CB$;

and the angle $CB'B = CBB'$ (Ch. 23, VIII.) $= CAB$ (Ch. 16, VIII.);

$\therefore CBB' = CAB$; add to each CBA;

$CBB' + CBA = CAB + CBA = A + B$; but

$CBB' + CBA = 180°$;

therefore $A + B = 180°$.

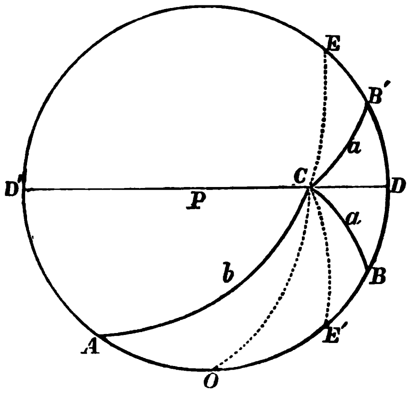
2. Suppose $a + b < 180°$;

then $A + B < 180°$.

Then C cannot be the pole of the arc ABE; for if

C were the pole, a and b would each be 90° (Ch. Art 37, VIII.); that is, $a+b$ would equal 180°, which is contrary to the hypothesis.

If $a=b$, CD being the shorter perpendicular from C upon BB', a and b are both greater than CD (Art. 132); therefore B and B' are each $< 90°$ (Art. 108), if CB' and CB are a and b; therefore their sum is less than 180°.

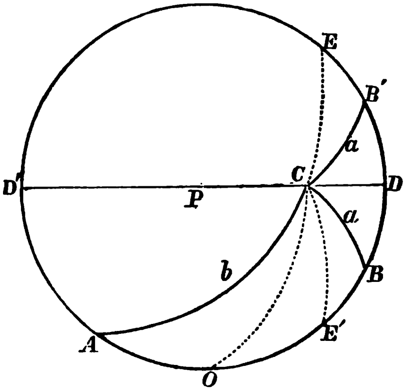


Suppose now a and b unequal, and $a<b$. Produce AC to meet the circumference of the base of the hemisphere at E. From C draw the arc $CE'=CE$. Then $b+CE=180°$ (Ch. Art. 32, VIII.).

$$\therefore (a+b) < (b+CE);\ a < CE;$$
$$\text{also } a < CE'.$$

From C two arcs, each equal to a, may be drawn to EDE', one above CD, between CD and CE, and the other below CD, between CD and CE' (Art. 132), CD

being the shorter perpendicular* from C upon the circumference of the base of the hemisphere.



Suppose first that CB is taken as a.
Now $CB < CE$, as just proved;
therefore the angle $CEB < CBE$ (Ch. 26, VIII.).
But $CEB = CAB$ (Ch. 16, VIII.);
therefore $CAB < CBE$; add CBA to each;
$CAB + CBA$ are less than $CBE + CBA$;

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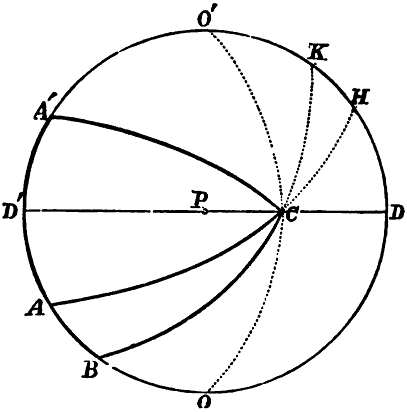
* CD is assumed as the shorter perpendicular; but if proof that it is the shorter perpendicular were required, take O, on the circumference of the base, as the pole of PCD and draw the arc CO. CO is an arc of 90° (Ch. Art. 37, VIII.). C has been proved not to be the pole of the base; therefore the proposition of Art. 132 applies, and the shorter perpendicular lies to the right of CO. But again, $a + b$ by hypothesis is less than 180°, and $a < b$, therefore a is $< 90°$ and $< CO$, and must lie to the right of CO. But the perpendicular is the least line from C to the circumference $OE'BD$, and is therefore less than a, or CB, and must lie to the right of a, or CB.

If a is the perpendicular itself the proposition is still true, for it is then $< 90°$ (Art. 108), and $D = 90°$.

but $CBE + CBA = 180°$;
therefore $CAB + CBA < 180°$, or
$$A + B < 180°.$$
Next suppose CB' is taken as a.
Since CD is the shorter perpendicular,
$CB'A < 90°$; also $CAB' < 90°$ (Art. 134);
therefore their sum $CB'A + CAB' < 180°$.

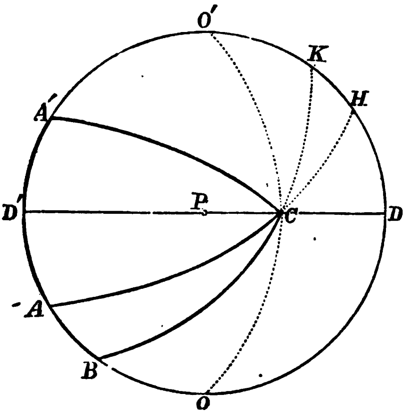


3. Let $a + b > 180°$; then also
$$A + B > 180°.$$

Now, if C were the pole of AB, a and b would each equal 90° (Ch. Art. 37, VIII.), and $a + b$ would equal 180°, which is contrary to the hypothesis; therefore C is not the pole of AB.

If $a = b$, a would lie on one side of CD' and b on the other side (Art. 132). Now, if we take O and O' on the circumference of the base, each at the distance of a quadrant from C, since a and b (if $a = b$) are each greater than 90°, a and b must each meet the circumference $OD'O'$ to the left of O and O' (Art. 132); therefore in this case the angles A and B are each obtuse (Art. 134), and their sum is greater than 180°, since CD' is the longer perpendicular from C.*

If a and b are unequal, then one of them, as a, might be below CD', and the other, as b, might occupy one of two positions (Art. 132), as CA or CA', one below CD' and the other above CD'.



If a be below CD', and b above CD', then $A'CB$ will be the triangle to be considered. $CA'B$ and CBA' are each obtuse in this case (Art. 134), and therefore $CA'B + CBA' > 180°$.

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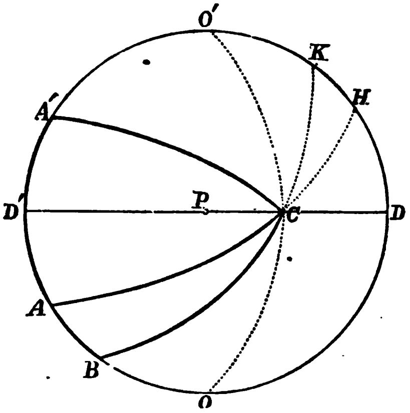
* This may be proved by a method similar to that employed in the note for 2; or A and B may be proved obtuse directly, thus:

$CD' > CB$.·. $B > D'$ (Ch. 26, VIII.); i. e., $B > 90°$; in the same way the equal angle is $> 90°$; .·. $A + B > 180°$.

If a and b are both below CD', then CAB is the triangle to be considered.

Produce AC and BC to meet the circumference of the base of the hemisphere again at H and K. Then ACH and BCK are semicircumferences (Ch. Art. 32, VIII.), and each equals 180°.



$\therefore a + CK = 180^\circ$ and $b + CH = 180^\circ$, or
$a + b + CH + CK = 360^\circ$, but $a + b > 180^\circ$.
$\therefore CH + CK < 180^\circ$;
therefore by the 2d case
(1) $CHK + CKH < 180^\circ$,
but $CHK = CAK = 180^\circ - CAB = 180^\circ - A$;
also $CKH = CBH$ (Ch. 16, VIII.) $= 180^\circ - B$.
Substituting these values in (1)
$360^\circ - (A + B) < 180^\circ$;
therefore $A + B > 180^\circ$.

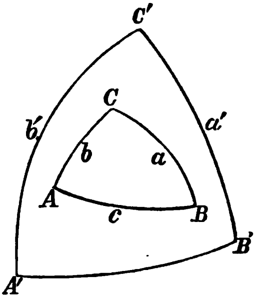
137. The converse of the preceding article is true, that is, the SUM of TWO SIDES of a spherical triangle is

180°, if the SUM of the OPPOSITE ANGLES is 180°; is LESS than 180°, if the SUM of the OPPOSITE ANGLES is LESS than 180°; is GREATER than 180°, if the SUM of the OPPOSITE ANGLES is greater than 180°.

Let two angles of a spherical triangle be denoted by A and B, and the sides opposite them by a and b respectively.

1. If $A+B=180°$, $a+b=180°$.

Let A' and B' be the corresponding angles of the polar triangle, and a' and b' the sides opposite these.



Then $A=180°-a'$, $B=180°-b'$,

$A'=180°-a$, $B'=180°-b$ (Ch. 18, VIII.);

$\therefore$ substituting for A and B their equivalents in 1,

$360°-(a'+b')=180°$; or $a'+b'=180°$;

consequently $A'+B'=180°$ (Art. 136).

Substituting values for A' and B',

$360°-(a+b)=180°$, or $a+b=180°$.

2. If $A+B<180°$, $a+b<180°$;

for, passing to the polar triangle,

$360°-(a'+b')<180°$, or $a'+b'>180°$;

consequently $A'+B'>180°$ (Art. 136); that is,

$360°-(a+b)>180°$, or $a+b<180°$.

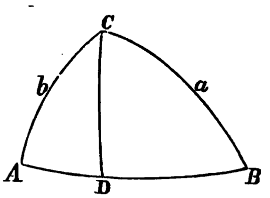
3. If $A+B>180°$, $a+b>180°$;
for, passing to the polar triangle,

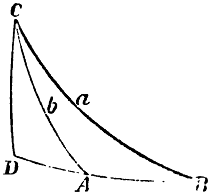
$360°-(a'+b')>180°$, or $a'+b'<180°$;

consequently $A'+B'<180°$ (Art. 136); that is,

$360°-(a+b)<180°$, or $a+b>180°$.

138. *The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*





Thus let ABC be any spherical triangle, of which the angles are A, B, and C, and the sides opposite them are a, b, and c, respectively. Then

$$\frac{\sin. a}{\sin. b}=\frac{\sin. A}{\sin. B}.$$

From C let the arc CD be drawn perpendicular to the side c, or (in the right-hand figure) to c produced, meeting it in the point D. Then ABC is equal to the sum or to the difference of two right-angled triangles.

In the triangle ACD, taking CD as a *middle* part and $90°-b$, $90°-CAD$ as *opposite* parts,

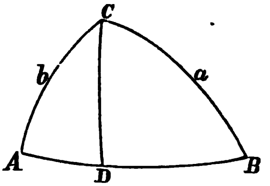
$$\begin{aligned}\sin. CD &= \sin. b \sin. CAD \text{ (Art. 115)}\\ &= \sin. b \sin. A \text{ (Art. 46).}\end{aligned}$$

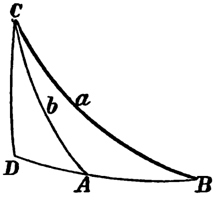
In the triangle BCD, also taking CD as a *middle* part and $90°-a$, $90°-B$ as *opposite* parts,

$\sin. CD = \sin. a \sin. B$;

$\therefore \sin. a \sin. B = \sin. b \sin. A$, or

$$(a)\ \frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B}.$$





In a similar manner it can be proved

$$(b)\ \frac{\sin. a}{\sin. c} = \frac{\sin. A}{\sin. C}.$$

From (a) by *alternation* we have

$$\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b}.$$

From (b) by *alternation*

$$\frac{\sin. A}{\sin. a} = \frac{\sin. C}{\sin. c};\ \text{therefore}$$

$$(c)\ \frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c}.$$

If the perpendicular coincided with the side a, B would be a right angle, and we should have from (a)

$$\sin. a = \sin. b \sin. A \text{ (Art. 35)}$$

an equation already established (Art. 109).

139. Suppose a to be $> b$, then A is $> B$ (Ch. 26, VIII.).

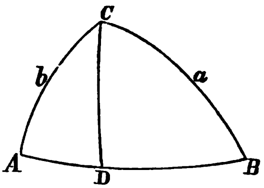
$$\frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B} \text{ (Art. 138);}$$

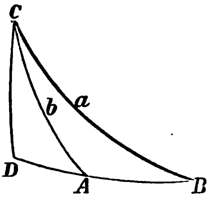
$$\frac{\sin. a + \sin. b}{\sin. a - \sin. b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B} \text{ (Ch. Art. 10, III.);}$$

$$\frac{\tan. \frac{1}{2}(a+b)}{\tan. \frac{1}{2}(a-b)} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)} \text{ ((a) Art. 71).}$$

140. If an arc of a great circle be drawn from one of the angles of a spherical triangle perpendicular to the opposite side, or to the opposite side produced, then

(1) *The* SINES *of the* SEGMENTS *of the* SIDE, *on which the perpendicular falls, will be proportional to the* NUMERICAL COTANGENTS *of the* ADJACENT ANGLES.





Thus, in the triangle ABC, if the perpendicular CD is drawn from C to c, or c produced,

$$\frac{\sin. AD}{\sin. BD} = \frac{\cot. A}{\cot. B}.$$

For in the triangle ACD, taking AD as a *middle* part and $90° - CAD$, CD, as *adjacent* parts,

(*a*) $\sin. AD = \cot. CAD \tan. CD$ (Art. 115);

and in triangle BCD, taking BD as a *middle* part and CD, $90° - B$, as *adjacent* parts,

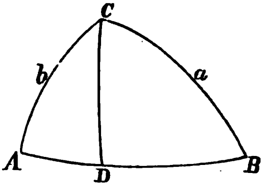
(*b*) $\sin. BD = \cot. B \tan. CD$;

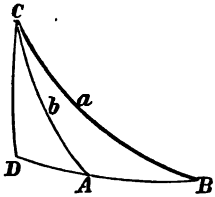
dividing equation (*a*) by equation (*b*),

$$(c)\ \frac{\sin.\ AD}{\sin.\ BD} = \frac{\cot.\ CAD}{\cot.\ B} = \frac{\cot.\ A}{\cot.\ B}\ \text{(Art. 46)},$$

cot. A being the numerical value of cot. A.

If the sign as well as the numerical value of cot. A be taken into account, equation (c) for the left-hand





figure, or in the case in which the perpendicular falls within the triangle, would be unaltered; but for the right-hand figure, or in the case in which the perpendicular falls without, (c) would become

$$(d)\ \frac{\sin.\ AD}{\sin.\ BD} = \frac{-\cot.\ A}{\cot.\ B}.$$

(2) *The* COSINES *of the* SEGMENTS *of the* SIDE *will be proportional to the* COSINES *of the* SIDES ADJACENT *to them.*

In triangle ACD, taking $90°-b$ as a *middle* part, and CD, AD as *opposite* parts,

$$\cos.\ AD \cos.\ CD = \cos.\ b \text{ (Art. 115)};$$

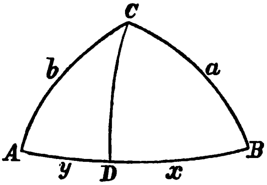
and in triangle BCD, taking $90°-a$ as a *middle* part, and BD, CD as opposite parts,

$$\cos.\ BD \cos.\ CD = \cos.\ a;$$

dividing the first of these equations by the second,

$$(e)\ \frac{\cos.\ AD}{\cos.\ BD} = \frac{\cos.\ b}{\cos.\ a}.$$

141. Let ABC be a spherical triangle, and from C let an arc of a great circle, CD, be drawn perpendicular to the opposite side.



Let the side a be $>b$, then is $A>B$ (Ch. 26, VIII.). Denote the segments BD and AD by x and y, respectively.

1. Let the perpendicular fall within the triangle.

$$\text{Then, tan. } \tfrac{1}{2}(x-y) = \frac{\sin.(A-B)}{\sin.(A+B)} \tan. \tfrac{1}{2} c.$$

$$\text{For } \frac{\sin. y}{\sin. x} = \frac{\cot. A}{\cot. B} \text{ ((1) Art. 140)}$$

$$= \frac{\tan. B}{\tan. A} \text{ ((}f\text{) Art. 64);}$$

$$\frac{\sin. x + \sin. y}{\sin. x - \sin. y} = \frac{\tan. A + \tan. B}{\tan. A - \tan. B} \text{ (Ch. Art. 10, III.);}$$

$$= \frac{\dfrac{\sin. A}{\cos. A} + \dfrac{\sin. B}{\cos. B}}{\dfrac{\sin. A}{\cos. A} - \dfrac{\sin. B}{\cos. B}} \text{ ((}b\text{) Art. 64);}$$

therefore $\dfrac{\tan. \frac{1}{2}(x+y)}{\tan. \frac{1}{2}(x-y)} = \dfrac{\sin.(A+B)}{\sin.(A-B)}$ ((*a*) Art. 71, (*a*) Art. 67, (*a*) Art. 68); or

$$\tan. \tfrac{1}{2}(x-y) = \frac{\sin.(A-B)}{\sin.(A+B)} \tan. \tfrac{1}{2} c,\text{* since } x+y=c.$$

---

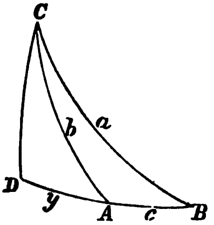
* If a is $>b$ and a and b are each $>90°$, A and B are also $>90°$ (Art. 136). Then y is $>x$ (Art. 132), and the equation above will be

$$\tan. \tfrac{1}{2}(y-x) = \frac{\sin.(B-A)}{\sin.(B+A)} \tan. \tfrac{1}{2} c.$$

The last equation may be written

$$(a)\ \tan.\tfrac{1}{2}(x-y)=\frac{\sin.\frac{1}{2}(A-B)\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)}\tan.\tfrac{1}{2}c$$

((*a*) Art. 73).



2. Let the perpendicular fall without the triangle.

BD and *AD*, denoted by x and y, are external segments.

$$\text{Then }\tan.\tfrac{1}{2}(x+y)=\frac{\sin.(A-B)}{\sin.(A+B)}\tan.\tfrac{1}{2}c.$$

$$\text{For }\frac{\sin.y}{\sin.x}=\frac{-\cot.A}{\cot.B}\ ((d)\text{ Art. }140)$$

$$=\frac{\cot.CAD}{\cot.B}\ (\text{Art. }46)=\frac{\tan.B}{\tan.CAD}((f)\text{ Art. }64);$$

$$\frac{\sin.x+\sin.y}{\sin.x-\sin.y}=\frac{\tan.CAD+\tan.B}{\tan.CAD-\tan.B}\ (\text{Ch. Art. }10,\text{ III.})$$

$$=\frac{\dfrac{\sin.CAD}{\cos.CAD}+\dfrac{\sin.B}{\cos.B}}{\dfrac{\sin.CAD}{\cos.CAD}-\dfrac{\sin.B}{\cos.B}}\ ((b)\text{ Art. }64);$$

But B is $<A$ (Ch. 26, VIII.), therefore $B-A$ is a negative quantity, and sin. $(B-A)=-$sin. $(A-B)$ (Art. 95).

Also tan. $\frac{1}{2}(y-x)=-$ tan. $\frac{1}{2}(x-y)$ (Art. 95).

Substituting these values our new equation becomes like the equation above—

$$\tan.\tfrac{1}{2}(x-y)=\frac{\sin.(A-B)}{\sin.(A+B)}\tan.\tfrac{1}{2}c.$$

therefore $\frac{\tan.\ \frac{1}{2}\ (x+y)}{\tan.\ \frac{1}{2}\ (x-y)}=\frac{\sin.\ (CAD+B)}{\sin.\ (CAD-B)}$ ((*a*) Art. 71), (*a*) Art. 67, (*a*) Art. 68).

Now, $CAD=180°-A$, and substituting the value of CAD in the right-hand member of the last equation,

$$\frac{\tan.\ \frac{1}{2}\ (x+y)}{\tan.\ \frac{1}{2}\ (x-y)}=\frac{\sin.\ \{180°-(A-B)\}}{\sin.\ \{180°-(A+B)\}}$$

$$=\frac{\sin.\ (A-B)}{\sin.\ (A+B)}\text{, or since } x-y=c,$$

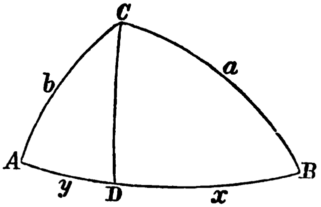
$$\tan.\ \tfrac{1}{2}\ (x+y)=\frac{\sin.\ (A-B)}{\sin.\ (A+B)}\ \tan.\ \tfrac{1}{2}\ c.$$

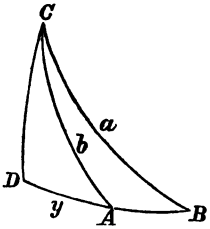
The last equation may be written

$$(b)\ \tan.\ \tfrac{1}{2}\ (x+y)=\frac{\sin.\ \frac{1}{2}\ (A-B)\ \cos.\ \frac{1}{2}\ (A-B)}{\sin.\ \frac{1}{2}\ (A\underset{+}{-}B)\ \cos.\ \frac{1}{2}\ (A+B)}\ \tan.\ \tfrac{1}{2}\ c$$

((*a*) Art. 73).

142. From the angle, C, of a spherical triangle, let an arc of a great circle, CD, be drawn perpendicular to the opposite side or opposite side produced.



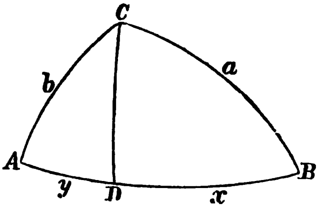


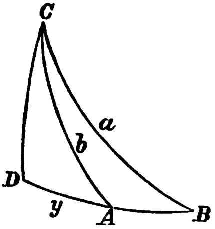
Suppose $a>b$, then is $A>B$ (Ch. 26, VIII.).

Denote the segments of the base by x and y, respectively.

Then $\tan. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a-b)$
$= \tan. \frac{1}{2}(x+y) \tan. \frac{1}{2}(x-y)$.

$$\frac{\cos. b}{\cos. a} = \frac{\cos. y}{\cos. x} \text{ ((2) Art. 140),}$$





$$\frac{\cos. b - \cos. a}{\cos. b + \cos. a} = \frac{\cos. y - \cos. x}{\cos. y + \cos. x} \text{ (Ch. Art. 10, III.), or}$$

$\tan. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a-b) = \tan. \frac{1}{2}(x+y) \tan. \frac{1}{2}(x-y)$ ((*b*) Art. 71, (*f*) Art. 64).

In the left-hand figure $x+y=c$;
in the right-hand figure $x-y=c$; therefore

(*a*) when the perpendicular falls within the triangle
$\tan. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a-b) = \tan. \frac{1}{2}c \tan. \frac{1}{2}(x-y)$; *

(*b*) when the perpendicular falls without the triangle,
$\tan. \frac{1}{2}(a+b) \tan. \frac{1}{2}(a-b) = \tan. \frac{1}{2}(x+y) \tan. \frac{1}{2}c$.

---

* When a is $>b$ and a and b are each $>90°$, y is $>x$ (Art. 132), and equation (*a*) above would become

$$\tan. \tfrac{1}{2}(a+b) \tan. \tfrac{1}{2}(b-a) = \tan. \tfrac{1}{2}c \tan. \tfrac{1}{2}(y-x).$$

But $\frac{1}{2}(b-a)$ is a negative quantity,

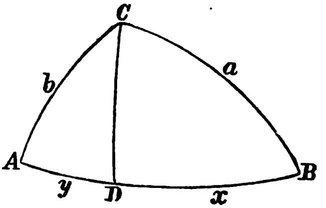
$$\tan. \tfrac{1}{2}(b-a) = -\tan. \tfrac{1}{2}(a-b) \text{ (Art. 95);}$$

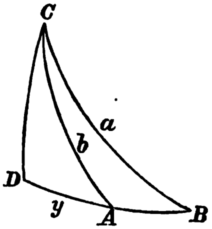
$$\text{also } \tan. \tfrac{1}{2}(y-x) = -\tan. \tfrac{1}{2}(x-y).$$

Substituting these values our new equation becomes like the equation above—

$$\tan. \tfrac{1}{2}(a+b) \tan. \tfrac{1}{2}(a-b) = \tan. \tfrac{1}{2}c \tan. \tfrac{1}{2}(x-y).$$

143. In a spherical triangle, the TANGENT of HALF the SUM of TWO SIDES *equals* the RATIO *of the* COSINE *of* HALF *the* DIFFERENCE *to the* COSINE *of* HALF *the* SUM *of the* OPPOSITE ANGLES, *multiplied by the* TANGENT *of* HALF *the* THIRD SIDE; *and the* TANGENT *of* HALF *the* DIFFERENCE *of* TWO SIDES *equals the* RATIO *of the* SINE *of* HALF *the* DIFFERENCE *to the* SINE *of* HALF *the* SUM *of the* OPPOSITE ANGLES, *multiplied by the* TANGENT *of* HALF *the* THIRD SIDE.





Thus, a, b, and c denoting the sides of a spherical triangle, and A, B, and C denoting the angles respectively opposite these, a also being greater than b,

$$(a)\ \tan.\ \tfrac{1}{2}(a+b)=\frac{\cos.\ \frac{1}{2}(A-B)}{\cos.\ \frac{1}{2}(A+B)}\tan.\ \tfrac{1}{2}c.$$

$$(b)\ \tan.\ \tfrac{1}{2}(a-b)=\frac{\sin.\ \frac{1}{2}(A-B)}{\sin.\ \frac{1}{2}(A+B)}\tan.\ \tfrac{1}{2}c.$$

1. When a perpendicular, drawn to the third side from the opposite angle, falls *within* the triangle, i. e., when A and B are both acute or both obtuse (Art. 135).

Denote the segments BD and AD by x and y, respectively.

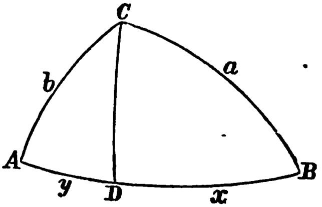
$$(1)\ \frac{\tan.\tfrac{1}{2}(a+b)}{\tan.\tfrac{1}{2}(a-b)}=\frac{\tan.\tfrac{1}{2}(A+B)}{\tan.\tfrac{1}{2}(A-B)}\ (\text{Art. }139).$$

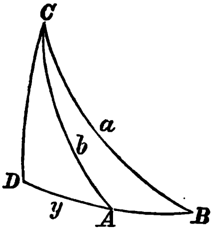
$$(2)\ \tan.\tfrac{1}{2}(x-y)=\frac{\sin.\tfrac{1}{2}(A-B)\cos.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A+B)}\tan.\tfrac{1}{2}c$$

((*a*) Art. 141).

$$(3)\ \frac{\tan.\tfrac{1}{2}(a+b)\tan.\tfrac{1}{2}(a-b)}{\tan.\tfrac{1}{2}(x-y)}=\tan.\tfrac{1}{2}c\ ((a)\ \text{Art. }142).$$

Multiply equations (1), (2) and (3) together, member by member, cancel like terms in numerator and denominator of the resulting fractions, and extract the square root of each member of the final equation, and





$$(4)\ \tan.\tfrac{1}{2}(a+b)=\frac{\cos.\tfrac{1}{2}(A-B)}{\cos.\tfrac{1}{2}(A+B)}\tan.\tfrac{1}{2}c.$$

Now equation (1) may be written

$$\frac{\tan.\tfrac{1}{2}(a-b)}{\tan.\tfrac{1}{2}(a+b)}=\frac{\tan.\tfrac{1}{2}(A-B)}{\tan.\tfrac{1}{2}(A+B)}.$$

Multiply the equation in this form by equation (4) and

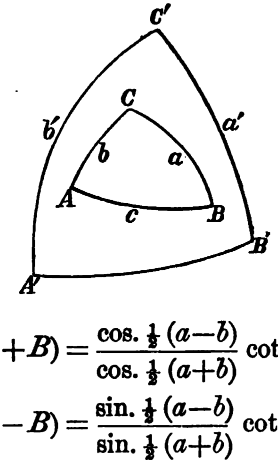
$$(5)\ \tan.\tfrac{1}{2}(a-b)=\frac{\sin.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}(A+B)}\tan.\tfrac{1}{2}c.$$

2. When the perpendicular falls *without* the triangle, i. e., when *A* is obtuse and *B* acute.

The proof is the same except that in equations (2) and (3) we have tan. $\frac{1}{2}(x+y)$ in place of tan. $\frac{1}{2}(x-y)$ ((*b*) Art. 141, and (*b*) Art. 142).

144. In a spherical triangle, *the* TANGENT *of* HALF *the* SUM *of two* ANGLES *equals the* RATIO *of the* COSINE *of* HALF *the* DIFFERENCE *to the* COSINE *of* HALF *the* SUM *of the* OPPOSITE SIDES, *multiplied by the* COTANGENT *of* HALF *the* THIRD ANGLE; *and the* TANGENT *of* HALF *the* DIFFERENCE *of* TWO ANGLES *equals the* RATIO *of the* SINE *of* HALF *the* DIFFERENCE *to the* SINE *of* HALF *the* SUM *of the* OPPOSITE SIDES, *multiplied by the* COTANGENT *of* HALF *the* THIRD ANGLE.

Thus, in the triangle ABC, a being $>b$,



$$(a)\ \tan.\tfrac{1}{2}(A+B)=\frac{\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}C.$$

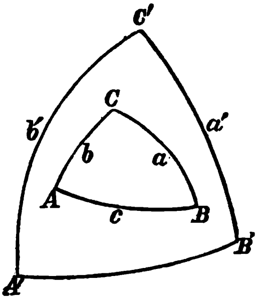
$$(b)\ \tan.\tfrac{1}{2}(A-B)=\frac{\sin.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}(a+b)}\cot.\tfrac{1}{2}C.$$

Let $A'B'C'$ be the polar triangle of ABC.
Since by hypothesis a is $> b$,
$180°-a$ is $<180°-b$; but $A'=180°-a$, and
$B'=180°-b$ (Ch. 18, VIII.);

therefore A' is $<B'$; consequently a' is $<b'$ (Ch. 26, VIII.).

(1) $\tan. \frac{1}{2}(b'+a') = \frac{\cos. \frac{1}{2}(B'-A')}{\cos. \frac{1}{2}(B'+A')} \tan. \frac{1}{2} c'$ (Art. 143);

but $a'=180°-A$; $b'=180°-A$; $c'=180°-C$;
$A'=180°-a$; $B'=180°-b$.



Substituting these values in (1)

$$\tan. \{180°-\tfrac{1}{2}(A+B)\} = \frac{\cos. \frac{1}{2}(a-b)}{\cos. \{180°-\frac{1}{2}(a+b)\}} \tan. \{90°-\tfrac{1}{2} C\}, \text{ or}$$

$\tan. \frac{1}{2}(A+B) = \frac{\cos. \frac{1}{2}(a-b)}{\cos. \frac{1}{2}(a+b)} \cot. \frac{1}{2} C$ (Art. 46, and Art. 5).

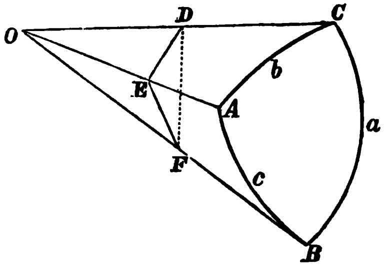
Again, $\tan. \frac{1}{2}(b'-a') = \frac{\sin. \frac{1}{2}(B'-A')}{\sin. \frac{1}{2}(B'+A')} \tan. \frac{1}{2} c$ (Art. 143).

Substituting for a', b', c', etc., their values

$$\tan. \tfrac{1}{2}(A-B) = \frac{\sin. \frac{1}{2}(a-b)}{\sin. \frac{1}{2}(a+b)} \cot. \tfrac{1}{2} C.$$

Equations (*a*) and (*b*) of this article, and equations (*a*) and (*b*) of the preceding article, written in the form of proportions, are called ***Napier's Analogies.***

145. *To find an expression for the* COSINE *of an* ANGLE IN TERMS *of the* SIDES *of a spherical triangle.*



Let *ABC* represent a triangle on the surface, corresponding to the triedral angle, *O*, at the centre of the sphere.

From *E*, any point in the edge of the diedral angle *OA*, let *DE* be drawn in the plane *OAC*, perpendicular to *OA*, and let *EF* be drawn in the plane *OAB*, also perpendicular to *OA*. Let *ED* meet *OC* in *D*, and let *EF* meet *OB* at *F*. Join *D* and *F* by the straight line *DF*. *DF* lies in the plane *OBC* (Euc. Def. 7, I. Ch. Def. 6, Int.).

DEF, the *measure* of the diedral angle whose edge is *OA* (Ch. Art. 39, and Art. 45, VI.), is also the *measure* of the angle *A* of the spherical triangle (Ch. 16, VIII.), and may be taken to represent that angle.

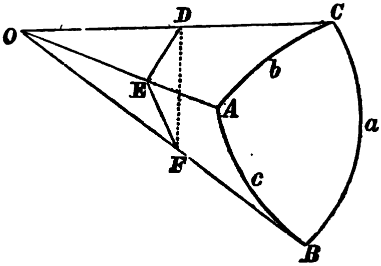
a, *b*, *c*, the sides of the spherical triangle, are the measures of the angles *BOC*, *COA*, *AOB*, respectively (Art. 102).

From triangle DEF

$$(1)\ \cos. A = \cos. DEF = \frac{DE^2 + EF^2 - DF^2}{2DE.EF}$$

((3) Art. 61).

Since DEO and EOF are triangles right-angled at



E, $DE^2 = DO^2 - OE^2$, and $EF^2 = OF^2 - OE^2$; substituting in (1) these values of DE^2 and EF^2,

$$\cos. A = \frac{DO^2 + OF^2 - DF^2 - 2OE^2}{2DE.EF}.$$

Divide numerator and denominator of the fraction by $2OD.OF$, and

$$\cos. A = \frac{\dfrac{DO^2 + OF^2 - DF^2}{2OD.OF} - \dfrac{OE}{OD} \times \dfrac{OE}{OF}}{\dfrac{DE}{OD} \times \dfrac{EF}{OF}}.$$

Now, $\dfrac{DO^2 + OF^2 - DF^2}{2OD.OF} = \cos. DOF = \cos. a$

((3) Art. 61);

$\dfrac{OE}{OD} = \cos. DOE = \cos. b$; $\dfrac{OE}{OF} = \cos. EOF = \cos. c$;

$\frac{DE}{OD} = \sin. b$; and $\frac{EF}{OF} = \sin. c$;

therefore

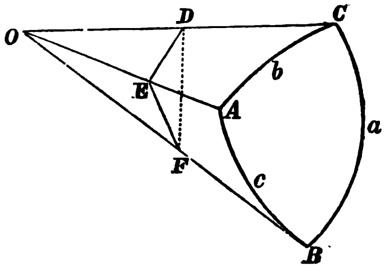
(a) $\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$.

In a similar manner it may be proved

(b) $\cos. B = \frac{\cos. b - \cos. c \cos. a}{\sin. c \sin. a}$;

(c) $\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b}$.

146. *To find an expression for the* TANGENT *of one* HALF *an* ANGLE, IN TERMS *of the* SIDES *of a spherical triangle.*



A, *B*, and *C* denoting the angles of the triangle, and *a*, *b*, *c* denoting the sides opposite these, *s* being $\frac{a+b+c}{2}$,

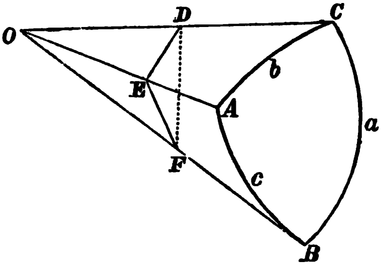
(a) $\tan. \frac{1}{2} A = \sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. s \sin. (s-a)}}$.

(b) $\tan. \frac{1}{2} B = \sqrt{\frac{\sin. (s-c) \sin. (s-a)}{\sin. s \sin. (s-b)}}$.

(*c*) $\tan. \tfrac{1}{2} C = \sqrt{\dfrac{\sin. (s-a) \sin. (s-b)}{\sin. s \sin. (s-c)}}$.

For

(*d*) $\cos. A = \dfrac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$ (Art. 145).



Subtract each member of the equation from 1.

$$1 - \cos. A = \frac{\sin. b \sin. c + \cos. b \cos. c - \cos. a}{\sin. b \sin. c}$$

$$= \frac{\cos. (b-c) - \cos. a}{\sin. b \sin. c} \quad ((b) \text{ Art. } 68)$$

$$= \frac{2 \sin. \dfrac{(a+b-c)}{2} \sin. \dfrac{(a+c-b)}{2}}{\sin. b \sin. c}$$

((*d*) Art. 70).

Now, let, (1), $\dfrac{a+b+c}{2} = s$; then, (2), $\dfrac{b+c-a}{2} = s-a$; (3), $\dfrac{a+c-b}{2} = s-b$; and, (4), $\dfrac{a+b-c}{2} = s-c$, substituting in the last equation the values of $\dfrac{a+c-b}{2}$ and $\dfrac{a+b-c}{2}$.

$$(e)\ 1-\cos. A=\frac{2\sin.(s-b)\sin.(s-c)}{\sin. b \sin. c}.$$

Add 1 to both members of equation (d), then

$$1+\cos. A=\frac{\cos. a-\cos. b\cos. c+\sin. b\sin. c}{\sin. b\sin. c}$$

$$=\frac{\cos. a-(\cos. b\cos. c-\sin. b\sin. c)}{\sin. b\sin. c};$$

$$\text{or } 1+\cos. A=\frac{\cos. a-\cos.(b+c)}{\sin. b\sin. c}\ ((b)\text{ Art. } 67)$$

$$=\frac{2\sin.\frac{(a+b+c)}{2}\sin.\frac{(b+c-a)}{2}}{\sin. b\sin. c}\ ((d)\text{ Art. } 70);$$

substituting values of $\frac{a+b+c}{2}$ and $\frac{b+c-a}{2}$,

$$(f)\ 1+\cos. A=\frac{2\sin. s\sin.(s-a)}{\sin. b\sin. c}.$$

Divide equation (e) by equation (f) and extract the square root of both members.

$$\sqrt{\frac{1-\cos. A}{1+\cos. A}}=\sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin. s\sin.(s-a)}};$$

or, since $\tan. \frac{1}{2}A=\sqrt{\frac{1-\cos. A}{1+\cos. A}}$ ((c) Art. 75),

$$\tan. \tfrac{1}{2}A=\sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin. s\sin.(s-a)}}.$$

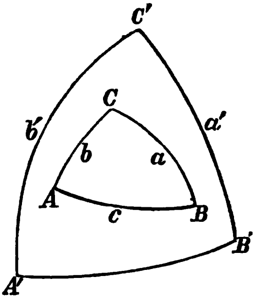
In a similar manner it can be proved

$$\tan. \tfrac{1}{2}B=\sqrt{\frac{\sin.(s-c)\sin.(s-a)}{\sin. s\sin.(s-b)}};$$

$$\tan.\ \tfrac{1}{2}\ C=\sqrt{\frac{\sin.\ (s-a)\ \sin.\ (s-b)}{\sin.\ s\ \sin.\ (s-c)}}.$$

147. To find an expression for the COSINE of a SIDE IN TERMS of the ANGLES of a spherical triangle.

Let ABC be a spherical triangle, A, B, C being the angles, a, b, c being the sides opposite these. Let $A'B'C'$ be the polar triangle of ABC, A', B', C' being its angles and a', b', c' being the sides opposite these.



$$(1)\ \cos.\ A'=\frac{\cos.\ a'-\cos.\ b'\ \cos.\ c'}{\sin.\ b'\ \sin.\ c'}\ ((a)\ \text{Art. 145}).$$

$$A'=180^\circ-a;\ a'=180^\circ-A;\ b'=180^\circ-B;$$
$$c'=180^\circ-C\ (\text{Ch. Arts. 69 and 70, VIII.});$$

substituting these values in (1)

$$-\cos.\ a=\frac{-\cos.\ A-\cos.\ B\cos.\ C}{\sin.\ B\ \sin.\ C}\ (\text{Art. 46}),\ \text{or}$$

$$(a)\ \cos.\ a=\frac{\cos.\ A+\cos.\ B\ \cos.\ C}{\sin.\ B\ \sin.\ C}.$$

In the same manner it may be proved

$$(b)\ \cos.\ b=\frac{\cos.\ B+\cos.\ C\ \cos.\ A}{\sin.\ C\sin.\ A};$$

$$(c)\ \cos.\ c = \frac{\cos.\ C + \cos.\ A \cos.\ B}{\sin.\ A \sin.\ B}.$$

148. *To find an expression for the* TANGENT *of* ONE HALF *a* SIDE, IN TERMS *of* THE ANGLES *of a spherical triangle.*

Let A, B, C denote the angles of the spherical triangle; let a, b, c denote the sides opposite these respectively, and let S denote $\frac{A+B+C}{2}$; then

$$(a)\ \tan.\ \tfrac{1}{2} a = \sqrt{\frac{-\cos.\ S \cos.\ (S-A)}{(\cos.\ S-B) \cos.\ (S-C)}}.$$

$$(b)\ \tan.\ \tfrac{1}{2} b = \sqrt{\frac{-\cos.\ S \cos.\ (S-B)}{\cos.\ (S-C) \cos.\ (S-A)}}.$$

$$(c)\ \tan.\ \tfrac{1}{2} c = \sqrt{\frac{-\cos.\ S \cos.\ (S-C)}{\cos.\ (S-A) \cos.\ (S-B)}}.$$

(*d*) For $\cos.\ a = \frac{\cos.\ A + \cos.\ B \cos.\ C}{\sin.\ B \sin.\ C}$.

Subtract each member of this equation from 1.

$$1 - \cos.\ a = \frac{-\{\cos.\ A + \cos.\ (B+C)\}}{\sin.\ B \sin.\ C}\ ((b)\ \text{Art. } 67)$$

$$= \frac{-2 \cos.\ \frac{A+B+C}{2} \cos.\ \frac{B+C-A}{2}}{\sin.\ B \sin.\ C.}\ ((c)\ \text{Art. } 70).$$

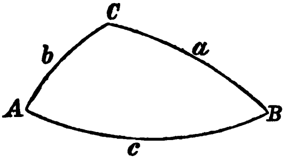
Now, let, (1), $S = \frac{A+B+C}{2}$; (2), $\frac{B+C-A}{2} = S - A$;

$$(3),\ \frac{A+C-B}{2}=S-B;\ (4),\ \frac{A+B-C}{2}=S-C.$$

Substituting values, from (1) and (2), in the last equation

$$(e)\ 1-\cos.\ a=\frac{-2\cos.\ S\cos.\ (S-A)}{\sin.\ B\sin.\ C}.$$

Again, adding 1 to both members of equation (*d*)



$$1+\cos.\ a=\frac{\cos.\ A+\cos.\ (B-C)}{\sin.\ B\sin.\ C}\ ((b)\ \text{Art. } 68)$$

$$=\frac{2\cos.\ \dfrac{A+B-C}{2}\cos.\ \dfrac{A+C-B}{2}}{\sin.\ B\sin.\ C},\ \text{or,}$$

$$(f)\ 1+\cos.\ a=\frac{2\cos.\ (S-B)\cos.\ (S-C)}{\sin.\ B\sin.\ C}$$

(see (3) and (4)).

Divide (*e*) by (*f*) and extract the square root.

$$\sqrt{\frac{1-\cos.\ a}{1+\cos.\ a}}=\sqrt{\frac{-\cos.\ S\cos.\ (S-A)}{\cos.\ (S-B)\cos.\ (S-C)}},\ \text{or,}$$

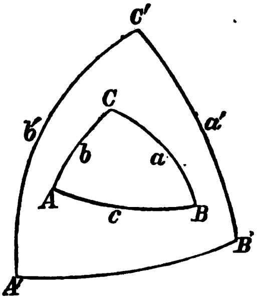
$$\tan.\ \tfrac{1}{2}a=\sqrt{\frac{-\cos.\ S\cos.\ (S-A)}{\cos.\ (S-B)\cos.\ (S-C)}}\ ((c)\ \text{Art. } 75).$$

In a similar manner it can be proved

$$\tan.\tfrac{1}{2}b = \sqrt{\frac{-\cos.S\cos.(S-B)}{\cos.(S-C)\cos.(S-A)}}$$

$$\tan.\tfrac{1}{2}c = \sqrt{\frac{-\cos.S\cos.(S-C)}{\cos.(S-A)\cos.(S-B)}}$$

149. In order that we may find real values for $\tan.\frac{1}{2}a$, $\tan.\frac{1}{2}b$, and $\tan.\frac{1}{2}c$, the quantity under the radical must be positive. This quantity will be positive if $\cos.S$ is negative (that is, if $-\cos.S$ is positive), and $\cos.(S-A)$, $\cos.(S-B)$, $\cos.(S-C)$ are all positive.



Now, $2S$ is $>180°$ and $<540°$ (Ch. 29, VIII.);

therefore S is $>90°$ and $<270°$;

consequently $\cos.S$ is negative (Art. 88).

Again, $\cos.(S-A)$, $\cos.(S-B)$, $\cos.(S-C)$ are all positive.

For, let $A'B'C'$ be the polar triangle of ABC.

Then $a'=180°-A$; $b'=180°-B$; $c'=180°-C$ (Ch. Arts. 69 and 70, VIII.).

But $a'<(b'+c')$ (Ch. 25, VIII.), or,

$(180°-A)<\{360°-(B+C)\}$.

$$\therefore \frac{B+C-A}{2} < \frac{180^\circ}{2} \text{ or } < 90^\circ.$$

But $\frac{B+C-A}{2} = S-A$ ((2) Art. 148);

therefore $S-A < 90^\circ$.

In the same way it can be proved that

$S-B < 90^\circ$ and $S-C < 90^\circ$;

therefore cos. $(S-A)$, cos. $(S-B)$, cos. $(S-C)$ are all positive (Art. 88).

150. $1-\cos. A = \frac{2 \sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}$ ((*e*) Art. 146);

or dividing each member of this equation by 2 and extracting the square root,

$$\sqrt{\frac{1-\cos. A}{2}} = \sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}}$$

But $\sin. \tfrac{1}{2} A = \sqrt{\frac{1-\cos. A}{2}}$ (Art. 59); therefore

$$(a) \quad \sin. \tfrac{1}{2} A = \sqrt{\frac{\sin. (s-b) \sin. (s-c)}{\sin. b \sin. c}}.$$

In a similar manner it may be proved

$$(b) \quad \sin. \tfrac{1}{2} B = \sqrt{\frac{\sin. (s-c) \sin. (s-a)}{\sin. c \sin. a}}.$$

$$(c) \quad \sin. \tfrac{1}{2} C = \sqrt{\frac{\sin. (s-a) \sin. (s-b)}{\sin. a \sin. b}}.$$

Also, $1+\cos. A = \frac{2 \sin. s \sin. (s-a)}{\sin. b \sin. c}$ ((*f*) Art. 146).

Divide each member of this equation by 2 and extract the square root.

$$\sqrt{\frac{1+\cos. A}{2}}=\sqrt{\frac{\sin. s \sin. (s-a)}{\sin. b \sin. c}}.$$

But $\cos. \frac{1}{2} A=\sqrt{\frac{1+\cos. A}{2}}$ (Art. 60); therefore

$$(d) \ \cos. \tfrac{1}{2} A=\sqrt{\frac{\sin. s \sin. (s-a)}{\sin. b \sin. c}}.$$

In a similar manner it can be proved

$$(e) \ \cos. \tfrac{1}{2} B=\sqrt{\frac{\sin. s \sin. (s-b)}{\sin. c \sin. a}}.$$

$$(f) \ \cos. \tfrac{1}{2} C=\sqrt{\frac{\sin. s \sin. (s-c)}{\sin. a \sin. b}}.$$

151. $1-\cos. a=\frac{-2 \cos. S \cos. (S-A)}{\sin. B \sin. C}$ ((*e*) Art. 148).

$$\sqrt{\frac{1-\cos. a}{2}}=\sqrt{\frac{-\cos. S \cos. (S-A)}{\sin. B \sin. C}}.$$

$$(a) \ \sin. \tfrac{1}{2} a=\sqrt{\frac{-\cos. S \cos. (S-A)}{\sin. B \sin. C}} \ \text{(Art. 59)}.$$

In a similar manner it can be proved

$$(b) \ \sin. \tfrac{1}{2} b=\sqrt{\frac{-\cos. S \cos. (S-B)}{\sin. C \sin. A}}.$$

$$(c) \ \sin. \tfrac{1}{2} c=\sqrt{\frac{-\cos. S \cos. (S-C)}{\sin. A \sin. B}}.$$

Again, $1+\cos. a = \dfrac{2\cos.(S-B)\cos.(S-C)}{\sin. B \sin. C}$ ((*f*) Art. 148).

$$\sqrt{\frac{1+\cos. a}{2}} = \sqrt{\frac{\cos.(S-B)\cos.(S-C)}{\sin. B\sin. C}}.$$

$$(d)\quad \cos.\tfrac{1}{2}a = \sqrt{\frac{\cos.(S-B)\cos.(S-C)}{\sin. B\sin. C}}.$$

In a similar manner it can be proved

$$(e)\quad \cos.\tfrac{1}{2}b = \sqrt{\frac{\cos.(S-C)\cos.(S-A)}{\sin. C\sin. A}};$$

$$(f)\quad \cos.\tfrac{1}{2}c = \sqrt{\frac{\cos.(S-A)\cos.(S-B)}{\sin. A\sin. B}}.$$

152. *Gauss's Equations.*

In a spherical triangle

$$(a)\quad \frac{\sin.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}C} = \frac{\cos.\frac{1}{2}(a-b)}{\cos.\frac{1}{2}c};$$

$$(b)\quad \frac{\cos.\frac{1}{2}(A+B)}{\sin.\frac{1}{2}C} = \frac{\cos.\frac{1}{2}(a+b)}{\cos.\frac{1}{2}c};$$

$$(c)\quad \frac{\sin.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}C} = \frac{\sin.\frac{1}{2}(a-b)}{\sin.\frac{1}{2}c};$$

$$(d)\quad \frac{\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}C} = \frac{\sin.\frac{1}{2}(a+b)}{\sin.\frac{1}{2}c}.$$

For $\dfrac{\sin. A}{\sin. B} = \dfrac{\sin. a}{\sin. b}$ ((*a*) Art. 138),

$$\frac{\sin. A+\sin. B}{\sin. A} = \frac{\sin. a+\sin. b}{\sin. a} \text{ (Ch. Art. 10, III.).}$$

$$\frac{2\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B)}{\sin.A}$$

$$=\frac{2\sin.\tfrac{1}{2}(a+b)\cos.\tfrac{1}{2}(a-b)}{\sin.a}\ ((a)\text{ Art. }73).$$

By alternation,

$$\frac{\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}(a+b)\cos.\tfrac{1}{2}(a-b)}=\frac{\sin.A}{\sin.a}=\frac{\sin.C}{\sin.c}$$

$$((c)\text{ Art. }138)=\frac{\sin.\tfrac{1}{2}C\cos.\tfrac{1}{2}C}{\sin.\tfrac{1}{2}c\cos.\tfrac{1}{2}c}$$

((*a*) Art. 73); therefore, again by alternation,

$$(1)\ \frac{\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}C\cos.\tfrac{1}{2}C}$$

$$=\frac{\sin.\tfrac{1}{2}(a+b)\cos.\tfrac{1}{2}(a-b)}{\sin.\tfrac{1}{2}c\cos.\tfrac{1}{2}c}.$$

$$(2)\ \frac{\cos.\tfrac{1}{2}(A+B)}{\cos.\tfrac{1}{2}(A-B)}=\frac{\tan.\tfrac{1}{2}c}{\tan.\tfrac{1}{2}(a+b)}\ ((a)\text{ Art. }143).$$

$$(3)\ \frac{\tan.\tfrac{1}{2}(A+B)}{\cot.\tfrac{1}{2}C}=\frac{\cos.\tfrac{1}{2}(a-b)}{\cos.\tfrac{1}{2}(a+b)}\ ((a)\text{ Art. }144).$$

Multiply together equations (1), (2), and (3), member by member, cancel like terms in numerator and denominator, and extract the square root of the result, and

$$(4)\ \frac{\sin.\tfrac{1}{2}(A+B)}{\cos.\tfrac{1}{2}C}=\frac{\cos.\tfrac{1}{2}(a-b)}{\cos.\tfrac{1}{2}c}\ (\text{equation }(a)).$$

Divide equation (1) by equation (4), and

$$(5)\ \frac{\cos.\tfrac{1}{2}(A-B)}{\sin.\tfrac{1}{2}C}=\frac{\sin.\tfrac{1}{2}(a+b)}{\sin.\tfrac{1}{2}c}\ (\text{equation }(d)).$$

Multiply equation (2) by equation (5).

(6) $$\frac{\cos.\ \frac{1}{2}(A+B)}{\sin.\ \frac{1}{2}C}=\frac{\cos.\ \frac{1}{2}(a+b)}{\cos.\ \frac{1}{2}c}\ \text{(equation }(b)\text{).}$$

Again, $$\frac{\tan.\ \frac{1}{2}(A-B)}{\cot.\ \frac{1}{2}C}=\frac{\sin.\ \frac{1}{2}(a-b)}{\sin.\ \frac{1}{2}(a+b)}\ ((b)\ \text{Art. } 144).$$

Multiply this last equation by equation (5), and

(7) $$\frac{\sin.\ \frac{1}{2}(A-B)}{\cos.\ \frac{1}{2}C}=\frac{\sin.\ \frac{1}{2}(a-b)}{\sin.\ \frac{1}{2}c}\ \text{(equation }(c)\text{).}$$

CHAPTER XV.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

Art. 153. Of the six parts of a spherical triangle (the three sides and the three angles), it is necessary to know *three* in order to *solve* the triangle.

To solve a *right-angled* triangle *three* parts were given, viz., the right angle and two other parts. (Arts. 116 and 119.)

To solve a *quadrantal* triangle *three* parts were given, viz., the quadrant and two other parts. (Arts. 129 and 131.)

154. There are six cases of the solution of oblique-angled spherical triangles, which may be classified under four heads, as the known parts are, in

(1) Case I.—The *three sides* ;

(2) Case II.—The *three angles* ;

(3) Two *sides* and an *angle*, under which head are

Case III.—Two *sides* and an *included angle*, and

Case IV.—Two *sides* and an angle *opposite* one of these sides ;

(4) Two *angles* and a *side*, under which head are

Case V.—Two *angles* and an *included side*, and

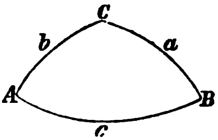
Case VI.—Two *angles* and a *side opposite* one of these angles.

155. Case I.—The *three sides* of an oblique-angled spherical triangle *being known, to solve the triangle.*

Use formulas (*a*), (*b*), and (*c*), Art. 146.

For check on the work use (*c*) Art. 138.

EXAMPLE. Suppose ABC to be a spherical triangle, of which $a=50°$, $b=40°$, $c=76°$.



$$\begin{aligned} a &= 50° \\ b &= 40° \\ c &= 76° \\ s = \frac{a+b+c}{2} &= \frac{166°}{2} = 83° \\ s-a &= 33° \\ s-b &= 43° \\ s-c &= 7° \end{aligned}$$

$$\text{Tan. } \tfrac{1}{2} A = \sqrt{\frac{\sin.\ (s-b) \sin.\ (s-c)}{\sin.\ s \sin.\ (s-a)}}$$

$$= \sqrt{\frac{\sin.\ 43° \sin.\ 7°}{\sin.\ 83° \sin.\ 33°}}.$$

Log. sin. 43° = 9.833783
Log. sin. 7° = 9.085894
Ar. co. log. sin. 83° = 0.003249
Ar. co. log. sin. 33° = 0.263891

2 | 19.186817

Log. tan. 21° 24′ 38.47″ = 9.5934085
∴ A = 42° 49′ 16.94″.

$$\text{Tan. } \tfrac{1}{2} B = \sqrt{\frac{\sin.\ (s-c) \sin.\ (s-a)}{\sin.\ s \sin.\ (s-b)}}$$

$$= \sqrt{\frac{\sin.\ 7° \sin.\ 33°}{\sin.\ 83° \sin.\ 43°}}$$

Log. sin. 7° = 9.085894
Log. sin. 33° = 9.736109
Ar. co. log. sin. 83° = 0.003249
Ar. co. log. sin. 43° = 0.166217

2 | 18.991469

Log. tan. 17° 23′ 14.18″ = 9.4957345
∴ B = 34° 46′ 28.36″.

$$\text{Tan. } \tfrac{1}{2} C = \sqrt{\frac{\sin.\ (s-a) \sin.\ (s-b)}{\sin.\ s \sin.\ (s-c)}}$$

$$= \sqrt{\frac{\sin.\ 33° \sin.\ 43°}{\sin.\ 83° \sin.\ 7°}}$$

Log. sin. 33° = 9.736109
Log. sin. 43° = 9.833783
Ar. co. log. sin. 83° = 0.003249
Ar. co. log. sin. 7° = 0.914106

2 | 20.487247

Log. tan. 60° 17′ 18.23″ = 10.2436235
∴ C = 120° 34′ 36.46″.

(*Check.*) $$\frac{\sin. A}{\sin. a} = \frac{\sin. B}{\sin. b} = \frac{\sin. C}{\sin. c}$$

Log. sin. 42° 49′ 16.94″ = 9.832327
Log. sin. 50° = 9.884254
$\bar{1}.948073$

Log. sin. 34° 46′ 28.36″ = 9.756140
Log. sin. 40° = 9.808067
$\bar{1}.948073$

Log. sin. 120° 34′ 36.46″ = 9.934977
Log. sin. 76° = 9.986904
$\bar{1}.948073$

Find the angles of a triangle when the sides are:

Example 1. 75°, 100°, 65°. *Ans.* 68° 8′ 54.4″, 108° 51′ 46.2″, 60° 33′ 33.3″.

2. 57°, 83°, 114°. *Ans.* 49° 3′ 20″, 63° 22′ 18.3″, 124° 38′ 8.6″.

3. 70°, 140°, 80°. *Ans.* 41° 22′ 18.7″, 153° 7′ 14.7″, 43° 50′ 32.1″.

4. 120°, 80°, 60°.

5. 100°, 69°, 51°.

156. Case II.—*The three angles* of an oblique-angled spherical triangle *being known, to solve the triangle.*

Use formulas (*a*), (*b*), and (*c*) of Art. 148.

For check on work use (*c*) Art. 138.

Example. Suppose the angles A, B, and C of a spherical triangle to be 130°, 60°, and 74° respectively, required the sides.

$$A = 130^\circ$$
$$B = 60^\circ$$
$$C = 74^\circ$$
$$S = \frac{A+B+C}{2} = \frac{264^\circ}{2} = 132^\circ$$
$$S - A = 2^\circ$$
$$S - B = 72^\circ$$
$$S - C = 58^\circ$$

$$\text{Tan.} \tfrac{1}{2} a = \sqrt{\frac{-\cos. S \cos. (S-A)}{\cos. (S-B) \cos. (S-C)}}$$
$$= \sqrt{\frac{-\cos. 132^\circ \cos. 2^\circ}{\cos. 72^\circ \cos. 58^\circ}}$$

Log. (− cos. 132°) = 9.825511
Log. cos. 2° = 9.999735
Ar. co. log. cos. 72° = 0.510018
Ar. co. log. cos. 58° = 0.275790

2 | 20.611054

Log. tan. 63° 40′ 17.55″ = 10.305527

$\therefore a = 127^\circ\ 20'\ 35.1''$.

$$\text{Tan.}\ \tfrac{1}{2}b=\sqrt{\frac{-\cos.\ S\ \cos.\ (S-B)}{\cos.\ (S-C)\cos.\ (S-A)}}=\sqrt{\frac{-\cos.\ 132^\circ\ \cos.\ 72^\circ}{\cos.\ 58^\circ\ \cos.\ 2^\circ}}$$

$$\begin{aligned}
\text{Log.}\ (-\cos.\ 132^\circ) &= 9.825511\\
\text{Log.}\ \cos.\ 72^\circ &= 9.489982\\
\text{Ar. co. log.}\ \cos.\ 58^\circ &= 0.275790\\
\text{Ar. co. log.}\ \cos.\ 2^\circ &= 0.000265\\
\hline
2\ |\ &\underline{19.591548}\\
\text{Log. tan.}\ 31^\circ\ 59'\ 56.81'' &= 9.795774
\end{aligned}$$

$\therefore\ b = 63^\circ\ 59'\ 53.62''$.

$$\text{Tan.}\ \tfrac{1}{2}c=\sqrt{\frac{-\cos.\ S\ \cos.\ (S-C)}{\cos.\ (S-A)\cos.\ (S-B)}}=\sqrt{\frac{-\cos.\ 132^\circ\ \cos.\ 58^\circ}{\cos.\ 2^\circ\ \cos.\ 72^\circ}}$$

$$\begin{aligned}
\text{Log.}\ (-\cos.\ 132^\circ) &= 9.825511\\
\text{Log.}\ \cos.\ 58^\circ &= 9.724210\\
\text{Ar. co. log.}\ \cos.\ 2^\circ &= 0.000265\\
\text{Ar. co. log.}\ \cos.\ 72^\circ &= 0.510018\\
\hline
2\ |\ &\underline{20.060004}\\
\text{Log. tan.}\ 46^\circ\ 58'\ 39.05'' &= 10.030002
\end{aligned}$$

$\therefore\ c = 93^\circ\ 57'\ 18.1''$.

(*Check.*) $$\frac{\sin.\ A}{\sin.\ a}=\frac{\sin.\ B}{\sin.\ b}=\frac{\sin.\ C}{\sin.\ c}$$

$$\begin{aligned}
\text{Log.}\ \sin.\ 130^\circ &= 9.884254\\
\text{Log.}\ \sin.\ 127^\circ\ 20'\ 35.1'' &= 9.900377\\
\hline
&\bar{1}.983877
\end{aligned}
\qquad
\begin{aligned}
\text{Log.}\ \sin.\ 60^\circ &= 9.937531\\
\text{Log.}\ \sin.\ 63^\circ\ 59'\ 53.6'' &= 9.953654\\
\hline
&\bar{1}.983877
\end{aligned}$$

$$\begin{aligned}
\text{Log.}\ \sin.\ 74^\circ &= 9.982842\\
\text{Log.}\ \sin.\ 93^\circ\ 57'\ 18.1'' &= 9.998965\\
\hline
&\bar{1}.983877
\end{aligned}$$

Find the sides of a triangle when the angles are:

Example 1. 64°, 75°, 117°. *Ans.* 68° 6′ 36.4″, 85° 43′ 1.4″, 113° 5′ 38.7″.

2. 136°, 72°, 102°. *Ans.* 147° 23′ 9.2″, 47° 33′ 14.4″, 130° 37′ 46.8″.

3. 125°, 75°, 100°.

4. 143° 3′, 119° 12′, 110° 35″. *Ans.* 140° 10′ $43\frac{1}{2}''$, 111° 34′ 17.4″. 85° 48′ 47.9″.

5. 129° 5′ 20″, 105° 8′, 142° 12′ 40″.

157. Case III.—*Two sides and an included angle of a spherical triangle being known, to solve the triangle.*

To find the angles: first, find the half sum and the half difference of the angles, opposite the given sides, by Art. 144; then add these quantities for the greater angle, and subtract the half difference from the half sum for the less angle.

To find the third side use (*a*) or (*b*) of Art. 143, or use Art. 138.

If the third side is found by Art. 143, for check on work use (*c*) Art. 138, otherwise use a formula of Art. 143.

REMARK.—When $\frac{1}{2}(a-b)$, and consequently $\frac{1}{2}(A-B)$, is a very small quantity, $\frac{1}{2}c$ is better found by (*a*) of Art. 143 than by (*b*) of same article, since the *cosines* of small angles differ from one another less than the sines (as will be seen from an inspection of the tables), and the effect of a small error in A or B would be greater if sin. $\frac{1}{2}(A-B)$ were used rather than cos. $\frac{1}{2}(A-B)$.

On the other hand, if $\frac{1}{2}(a+b)$, and consequently $\frac{1}{2}(A+B)$, is near 90°, to find $\frac{1}{2}c$ (*b*) Art. 143 should be preferred, since near 90° the sines differ less than the cosines.

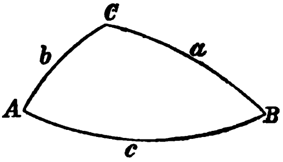
EXAMPLE. Suppose ABC to be a spherical triangle, of which the side $a=76^\circ$, $b=55^\circ\ 10'$, and the angle $C=125^\circ$.

$$\begin{aligned} a &= 76^\circ \\ b &= 55^\circ\ 10' \\ \hline \end{aligned}$$

$$\tfrac{1}{2}(a+b)=\frac{131^\circ\ 10'}{2}=65^\circ\ 35';$$

$$\tfrac{1}{2}(a-b)=\frac{20^\circ\ 50'}{2}=10^\circ\ 25';$$

$$\tfrac{1}{2}C=62^\circ\ 30'.$$



$$\tan.\ \tfrac{1}{2}(A+B)=\frac{\cos.\ \frac{1}{2}(a-b)}{\cos.\ \frac{1}{2}(a+b)}\cot.\ \tfrac{1}{2}C;$$

$$=\frac{\cos.\ 10^\circ\ 25'}{\cos.\ 65^\circ\ 35'}\cot.\ 62^\circ\ 30';$$

$$\begin{aligned} \text{Log. cos. } 10^\circ\ 25' &= 9.992783 \\ \text{Log. cot. } 62^\circ\ 30' &= 9.716477 \\ \text{Ar. co. log. cos. } 65^\circ\ 35' &= 0.383662 \\ \hline \text{Log. tan. } 51^\circ\ 4'\ 59.77'' &= 10.092922 \end{aligned}$$

$$\tan.\ \tfrac{1}{2}(A-B)=\frac{\sin.\ \frac{1}{2}(a-b)}{\sin.\ \frac{1}{2}(a+b)}\cot.\ \tfrac{1}{2}C;$$

$$=\frac{\sin.\ 10^\circ\ 25'}{\sin.\ 65^\circ\ 35'}\cot.\ 62^\circ\ 30';$$

$$\begin{aligned}
\text{Log. sin. } 10^\circ\ 25' &= 9.257211\\
\text{Log. cot. } 62^\circ\ 30' &= 9.716477\\
\text{Ar. co. log. sin. } 65^\circ\ 35' &= 0.040690\\
\hline
\text{Log. tan. } 5^\circ\ 54'\ 5.34'' &= 9.014378
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(A+B) &= 51^\circ\ \ 4'\ 59.77''\\
\tfrac{1}{2}(A-B) &= \ \ 5^\circ\ 54'\ \ 5.34''\\
\hline
A &= 56^\circ\ 59'\ \ 5.11''\\
B &= 45^\circ\ 10'\ 54.43''
\end{aligned}$$

$$\begin{aligned}
\tan.\ \tfrac{1}{2}c &= \frac{\cos.\ \frac{1}{2}(A+B)}{\cos.\ \frac{1}{2}(A-B)}\tan.\ \tfrac{1}{2}(a+b);\\
&= \frac{\cos.\ 51^\circ\ \ 4'\ 59.77''}{\cos.\ \ 5^\circ\ 54'\ \ 5.34''}\tan.\ 65^\circ\ 35'.
\end{aligned}$$

$$\begin{aligned}
\text{Log. cos. } 51^\circ\ \ 4'\ 59.77'' &= \ \ 9.798092\\
\text{Log. tan. } 65^\circ\ 35' &= 10.342972\\
\text{Ar. co. log. cos. } \ 5^\circ\ 54'\ \ 5.34'' &= \ \ 0.002308\\
\hline
\text{Log. tan. } 54^\circ\ 17'\ \ 24\tfrac{2}{3}'' &\ \ \ 10.143372\\
\therefore c = 108^\circ\ 34'\ \ 49\tfrac{1}{3}''
\end{aligned}$$

(*Check.*)
$$\frac{\sin.\ A}{\sin.\ a} = \frac{\sin.\ B}{\sin.\ b} = \frac{\sin.\ C}{\sin\ c}.$$

$$\begin{aligned}
\text{Log. sin. } 56^\circ\ 59'\ 5.11'' &= 9.923516\\
\text{Log. sin. } 76^\circ &= 9.986904\\
\hline
&\ \ \ \bar{1}.936612
\end{aligned}$$

$$\begin{aligned}
\text{Log. sin. } 45^\circ\ 10'\ 54.43'' &= 9.850858 & \text{Log. sin. } 125^\circ &= 9.913365\\
\text{Log. sin. } 55^\circ\ 10' &= 9.914246 & \text{Log. sin. } 108^\circ\ 34'\ 49\tfrac{1}{3}'' &= 9.976753\\
\hline
&\ \ \ \bar{1}.936612 & &\ \ \ \bar{1}.936612
\end{aligned}$$

Solve a spherical triangle when there are given the two sides and the included angle:

Example 1. Sides 112° 30′, 60° 15′; angle 35° 40′.

Ans. Angles 142° 36′ 17.7″, 34° 47′ 58.1″; side 62° 29′ 57.4″.

2. Sides 77° 41′, 54° 16′; angle 122° 13′.

Ans. Angles 59° 59′ 36.7″, 46° 0′ 49.2″; side 107° 21′ 2.7″.

3. Sides 140°, 113° 22′; angle 110° 16′.

Ans. Angles 142° 41′ 58.2″, 120° 3′ 59.9″; side 84° 17′ 40.3″

4. Sides 105°, 65°; angle 40°.

158. Case IV.—*Two sides and an angle opposite one of these sides being known, to solve the triangle.*

The angle opposite the other given side can be found by means of Art. 138.

The third side may be found by either of the formulas of Art. 143. Formula (*a*) should be preferred if $\frac{1}{2}(a-b)$ is a small quantity, but formula (*b*) if $\frac{1}{2}(a+b)$ is near 90° (see remark under preceding Art.).

The third angle may be found by either of the formulas of Art. 144. Formula (*a*) is to be preferred if $\frac{1}{2}(a-b)$ is a small quantity, but formula (*b*) if $\frac{1}{2}(a+b)$ is near 90° (on the principle given in remark of preceding Art.).

As check use Art. 138.

As the first required part is found from its sine, and as the sine of an acute angle and the sine of its supplement are the same, both in value and in sign, the part found may be either less than 90° or greater than 90° (Art. 46).

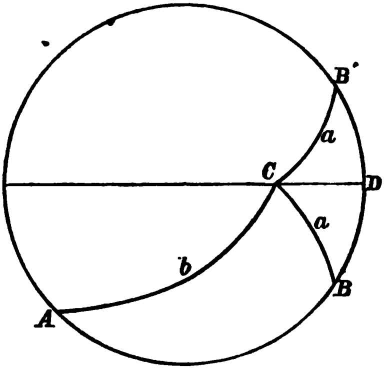
In some examples, therefore, there may be two triangles having the given parts, and in such examples we shall obtain two solutions, each of which will be correct.

This will be evident from the following figure.

Suppose the given parts to be a, b, and the angle A.

Let the triangle be projected upon the plane of the base of the hemisphere, of which the circumference contains the side c as a part of it. Let CD be the projection of the perpendicular from C (to the arc ABD).

From C two arcs can be drawn to the circumference ABB', each equal to a, one on each side of CD (Art. 132). Thus the triangles ABC and $AB'C$ each contain the angle A, the side b, and the side a.



There will *not always* be two solutions under this case. For instance, if $a+b=180°$, and A is given, B will be known, since $A+B=180°$ (Art. 136).

159. *Two sides and an angle opposite one of these being given, to determine whether the parts belong to one triangle or to two triangles;* that is, whether there should be *one* solution or *two* solutions.

Considering alone the sum of the given sides, we may bring all examples under three heads, as the *sum* of the given sides is:

1. *equal* to 180°;
2. *less* than 180°;
3. *greater* than 180°.

Under all these heads there will be *one* solution if the given sides are *equal* (Ch. 23, VIII.).

In what follows, therefore, we shall consider the sides *unequal*.

1. Let the sum of the given sides equal 180°.

There will be *one* solution, since the sum of the opposite angles must equal 180° (Art. 136).

Thus, suppose $a+b=180°$, and A is given, then $A+B=180°$; therefore $B=180°-A$.

2. Let the sum of the given sides be *less* than 180°.

Suppose $a+b<180°$, and A be the given angle:

(*a*) Let a be $>b$; then there will be *one* solution.

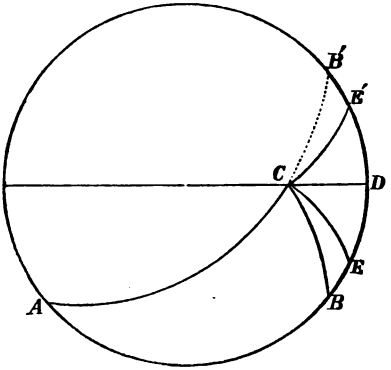
For suppose A to be $>90°$, then B must be $<90°$ (Art. 136); but suppose A to be $<90°$, then B must be $<90°$ (Ch. 26, VIII.).

(*b*) Let a be $<b$, and $A<90$; there will be *two* solutions, since B might have two values, one <90 and greater than A, or another $>90°$, and therefore $>A$ (Ch. 26, VIII.), and $A+B$ would still be $<180°$ (Art. 136).

Thus, suppose $a+b<180°$, and $A=35°$, B might be 50° or 130°, as with either value A is $<B$, and $A+B<180°$.

When $a+b<180°$, and a is $<b$, and A is $>90°$, no triangle can be formed, for B would also be >90 (Ch. 26, VIII.), and $A+B$ would be $>180°$, which is impossible (Art. 136).

The first and second classes of examples may be illustrated by the accompanying figure, which represents the surface of a hemisphere projected upon the plane of its base (Art. 133). CAB represents a spherical triangle. CD represents the perpendicular from C to the circumference



of the base of the hemisphere. ACB' represents a semicircumference (Ch. Art. 32, VIII.). CE and CE' represent two equal arcs. CB is an arc of a great circle, drawn from C, equal to CB' (Art. 132).

1. $a+b=180°=AC+CB$.
2. (a) $a+b<180°$; $(a>b,\ A\,(=E\,)>90°)=AC+CE$.
   $(a>b,\ A\,(=E')<90°)=AC+CE'$.
   (b) $a+b<180°$, $a<b$ and $A<90°=EC+CA$, or $E'C+CA$.

3. Let the sum of the given sides be greater than 180°.

(a) Suppose $a+b>180°$ and a is $<b$.

Now, whether A is $<90°$ or $>90°$, B must be $>90°$, and so there will be but *one* solution; for

If A is $<90°$, $A+B>180°$ (Art. 136), and therefore B must be $>90°$, and can have but one value, and

If A is $>90°$, B is $>A$ (Ch. 26, VIII.), and therefore B must be $>90°$, and can have but one value.

(b) If, now, a is $>b$ and $A>90°$, B may be either $<90°$ or $>90°$, for, with either value of B, $A+B$ might be $>180°$, and $A>B$ (Ch. 26, VIII.), so that in this case there would be *two* solutions.

Thus, suppose $a+b>180°$, $a>b$, and $A=110°$; B might be 75° or 105°. With either value of B, A is $>B$, $A+B>180°$.

When $a+b>180°$ and a is $>b$, and $A<90°$, no triangle is formed; for B must be $<A$, i. e., $<90°$, and $A+B$ would be $<180°$, which would be impossible.

The last case, (3), when $a+b>180°$, may also be illustrated by a figure.

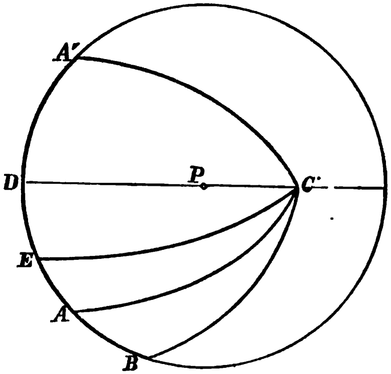
Let the hemisphere be projected upon the plane of its base (Art. 133).

CA, CA', CE, and CB are all arcs of great circles drawn from C. CA and CA' are equal. CD is the longer perpendicular from C upon the circumference of the base of the hemisphere.

$$(a)\ a+b>180°\ \left.\begin{array}{l}(a<b,\ A<90°)=CB+CA\\ (a<b,\ A>90°)=CB+CA'\end{array}\right\}\ \text{(Art. 132).}$$

(b) $a+b>180°$ $(a>b,\ A>90°)=CE+CA$ or $CE+CA'$.

Therefore, when two sides of a spherical triangle

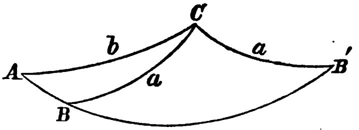


and an angle opposite one of them are given (considering only possible cases),

If the sum of the given sides is LESS *than* 180°, *and if the side opposite the given angle is* LESS *than the other given side* ((*b*) 2), or,

If the sum of the given sides is GREATER *than* 180°, *and if the side opposite the given angle is* GREATER *than the other given side* ((*b*) 3):

There will be TWO solutions; in all OTHER cases there will be ONE solution.



Example. Suppose $a=62° 40'$, $b=73° 13'$, and $A=44° 18'$.

As $a+b=135° 55' < 180°$, and as A is $<90°$, there are two solutions. (See rule.)

$$\text{Sin. } B=\frac{\text{sin. } b \text{ sin. } A}{\text{sin. } a}=\frac{\text{sin. } 73° 13' \text{ sin. } 44° 18'}{\text{sin. } 62° 42'}$$

Log. sin. $73^\circ\,18'$ $=9.981095$
Log. sin. $44^\circ\,18'$ $=9.844114$ $\qquad \tfrac{1}{2}(b+a)=67^\circ\,57'\,30''$
Ar. co. log. sin. $62^\circ\,42'$ $=0.051285$ $\qquad \tfrac{1}{2}(b-a)=\ 5^\circ\,15'\,30''$
Log. sin. $48^\circ\,48'\,20''=9.876494$

$B'=\ 48^\circ\,48'\,20''$ $\qquad \tfrac{1}{2}(B'+A)=46^\circ\,33'\,10''$ $\qquad \tfrac{1}{2}(B'-A)=\ 2^\circ\,15'\,10''$
$B=131^\circ\,11'\,40''$ $\qquad \tfrac{1}{2}(B+A)=87^\circ\,44'\,50''$ $\qquad \tfrac{1}{2}(B-A)=43^\circ\,26'\,50''$

Solving triangle $AB'C$,

$$\text{Tan. }\tfrac{1}{2}\,c=\frac{\cos.\ \frac{1}{2}\,(B'+A)}{\cos.\ \frac{1}{2}\,(B'-A)}\ \tan.\ \tfrac{1}{2}(a+b)=\frac{\cos.\ 46^\circ\,33'\,10''}{\cos.\ \ 2^\circ\,15'\,10''}\ \tan.\ 67^\circ\,57'\,30''$$

Log. cos. $46^\circ\,33'\,10''$ $=\ 9.837390$
Log. tan. $67^\circ\,57'\,30''$ $=10.392682$
Ar. co. log. cos. $2^\circ\,15'\,10''$ $=\ 0.000336$
Log. tan. $59^\circ\,31'\,55.62''=10.230408$
$c=AB'=119^\circ\ \ 3'\,51.25''$

$$\text{Cot. }\tfrac{1}{2}\,C=\frac{\cos.\ \frac{1}{2}\,(b+a)}{\cos.\ \frac{1}{2}\,(b-a)}\ \tan.\ \tfrac{1}{2}\,(A+B')=\frac{\cos.\ 67^\circ\,57'\,30''}{\cos.\ \ 5^\circ\,15'\,30''}\ \tan.\ 46^\circ\,33'\,10''$$

Log. cos. $67^\circ\,57'\,30''$ $=\ 9.574356$
Log. tan. $46^\circ\,33'\,10''$ $=10.023551$
Ar. co. log. cos. $5^\circ\,15'\,30''$ $=\ 0.001832$
Log. cot. $68^\circ\,18'\,14.26''=\ 9.599739$
$C=ACB'=136^\circ\,36'\,28.52''$

Solving triangle ABC,

$$\text{Tan. }\tfrac{1}{2}\,c=\frac{\sin.\ \frac{1}{2}\,(B+A)}{\sin.\ \frac{1}{2}\,(B-A)}\ \tan.\ \tfrac{1}{2}\,(b-a)=\frac{\sin.\ 87^\circ\,44'\,50''}{\sin.\ 43^\circ\,26'\,50''}\ \tan.\ 5^\circ\,15'\,30''$$

Log. sin. $87^\circ\,44'\,50''$ $=9.999664$
Log. tan. $5^\circ\,15'\,30''$ $=8.963947$
Ar. co. log. sin. $43^\circ\,26'\,50''$ $=0.162610$
Log. tan. $7^\circ\,37'\ \ 0.6''=9.126221$
$c=AB=15^\circ\,14'\ \ 1.25''$

$$\text{Cot. }\tfrac{1}{2}\,C=\frac{\sin.\ \frac{1}{2}\,(b+a)}{\sin.\ \frac{1}{2}\,(b-a)}\ \tan.\ \tfrac{1}{2}\,(B-A)=\frac{\sin.\ 67^\circ\,57'\,30''}{\sin.\ \ 5^\circ\,15'\,30''}\ \tan.43^\circ\,26'\,50''$$

Log. sin. $67^\circ\,57'\,30''$ $=\ 9.967038$
Log. tan. $43^\circ\,26'\,50''$ $=\ 9.976449$
Ar. co. log. sin. $5^\circ\,15'\,30''$ $=\ 1.037884$
Log. cot. $5^\circ\,57'\,32.79''=10.981371$
$C=ACB=11^\circ\,55'\ \ 5.58''$

(*Check.*) $$\frac{\text{sin. } B}{\text{sin. } b}=\frac{\text{sin. } ACB'}{\text{sin. } AB'}=\frac{\text{sin. } ACB}{\text{sin. } AB}$$

Log. sin. 48° 48′ 20″ $=9.876494$ Log. sin. 136° 36′ 28.52″ $=9.836948$
Log. sin. 73° 13′ $=9.981095$ Log. sin. 119° 3′ 51.25″ $=9.941549$
$\bar{1}.895399$ $\bar{1}.895399$

Log. sin. 11° 55′ 5.58″ $=9.314952$
Log. sin. 15° 14′ 1.25″ $=9.419553$
$\bar{1}.895399$

Solve a triangle when there are given:

EXAMPLE 1. Sides, 62° 14′, 50° 3′; angle opposite latter, 35° 33′.
Ans. Angles, 131° 25′ 9.6″, 42° 9′ 5.8″; side, 98° 36′ 12.1″;
or 11° 45′ 13.8″, 137° 50′ 54.2″; or 15° 34′ 49.7″.

2. Sides, 135° 10′, 115°; angle opposite first, 143°.
Ans. Angles, 50° 40′ $44\frac{4}{9}$″, 23° 13′ 13.6″; side, 27° 30′ 35.2″;
or 129° 19′ $15\frac{5}{9}$″, 121° 27′ or 91° 55′ 50.5″.

3. Sides, 137° 2′, 145°; angle opposite first, 151°.
Ans. Angles, 155° 55′ 16.8″, 137° 28′ 16.4″; side, 71° 51′ 40″.

4. Sides, 132° 10′, 63° 20′; angle opposite latter, 78° 20′.

160. CASE V.—*Two angles and an included side* of a spherical triangle *being known, to solve the triangle.*

To find the half sum and half difference of the sides, use formulas of Art. 143.

The sum of these differences will be the greater side; the difference of these differences will be the less side.

The third angle may be found by Art. 144.

Use formula (a) if $\frac{1}{2}(a-b)$ is very small; but formula (b) if $\frac{1}{2}(a+b)$ is near 90°.

As check on work use (c) Art. 138.

Thus, suppose the parts given are A, B, and c.

$$\left.\begin{aligned}\text{Tan. }\tfrac{1}{2}(a+b)&=\frac{\text{cos. }\frac{1}{2}(A-B)}{\text{cos. }\frac{1}{2}(A+B)}\text{ tan. }\tfrac{1}{2}c\\ \text{tan. }\tfrac{1}{2}(a-b)&=\frac{\text{sin. }\frac{1}{2}(A-B)}{\text{sin. }\frac{1}{2}(A+B)}\text{ tan. }\tfrac{1}{2}c\end{aligned}\right\}\text{ will give } a \text{ and } b.$$

$$\cot. \tfrac{1}{2}\, C = \frac{\cos. \frac{1}{2}\,(a+b)}{\cos. \frac{1}{2}\,(a-b)} \tan. \tfrac{1}{2}\,(A+B), \text{ or}$$

$$\cot. \tfrac{1}{2}\, C = \frac{\sin. \frac{1}{2}\,(a+b)}{\sin. \frac{1}{2}\,(a-b)} \tan. \tfrac{1}{2}\,(A-B), \text{ will give } C.$$

Solve a triangle when there are given:

EXAMPLE 1. Two angles, 33° 16′, 12° 14′; the included side, 49° 14′.
Ans. **Sides, 38° 14′ 16.8″, 13° 49′ 57.2″; angle, 137° 50′ 14.8″.**

2. Two angles, 140° 10′, 110° 2′; the included side, 125° 4′.
Ans. **Sides, 138° 37′ 58.8″, 75° 45′ 58.8″; angle, 127° 30′ 12.8″.**

3. Two angles, 78°, 56°; the included side, 59°.
Ans. **Sides, 61° 33′ 39.8″, 48° 10′ 59.3″; angle, 72° 27′ 31.8″.**

4. Two angles, 150°, 110°; the included side, 130°.

5. Two angles, 40°, 82 °; the included side, 75°.

161. CASE VI.—*Two angles and a side opposite one of the given angles* of an oblique-angled spherical triangle *being known, to solve the triangle.*

The side *opposite* the *other given* angle may be found by Art. 138.

The *third side* may then be found by Art. 143.

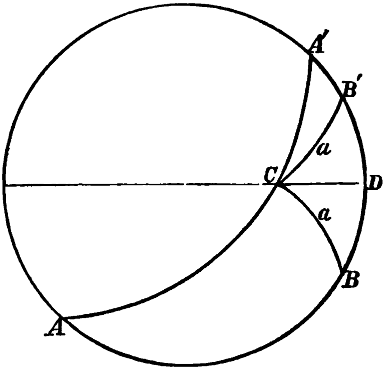
Use formula (*a*) if one half the difference of the given angles is a very small quantity, but use (*b*) if one half the sum of the given angles is near 90°. (See remark under Art. 157.)

The *third angle* may be found by Art. 144.

Use formula (*a*) if one half the difference of the sides opposite the given angles is a very small quantity, but use (*b*) if one half the sum of the same sides is near 90°.

As check on work use Art. 138.

As the first side is found by means of its sine, and as the sine of an angle and the sine of its supplement are the same (Art. 46), there may be two values given to the side, either of which will satisfy the conditions of the problem. Consequently there may be two triangles with the given parts, and in such a case two solutions are possible, both of which will be correct. This will be evident from the accompanying figure, in which the surface of a hemisphere is represented as projected upon the plane of its base (Art. 133).



Suppose the angles *CAB* and *CBA* and the side *CB* are the given parts. Let the arc *CA* be produced to meet the plane of the base of the hemisphere again at *A'*, so that *ACA'* is a semicircumference (Ch. Art. 32, VIII.). Suppose the given side *CB* is not equal to 90°; then *C* is not the pole of the base (Ch. Art. 37, VIII.). Since *C* is not the pole of the base, from *C* two arcs *CB* and *CB'* can be drawn to the base equal to one another (Art. 132). The angles *CBB'* and *CB'B* are equal (Ch. 23, VIII.); therefore their supplements *CBA* and *CB'A'* are equal. Also *A* is equal to *A'* (Ch. 16, VIII.).

Thus we have two triangles CAB and $CA'B'$, which have two angles in one equal respectively to two angles in the other, and the side opposite one of the equal angles the same in both. The side opposite the other given angle of one is the supplement of the side opposite the equal given angle of the other. Thus, CA opposite CBA is the supplement of CA', which is opposite $CB'A'$ (the equal of CBA).

162. *Two angles and a side opposite one of these being given, to determine whether these parts belong to one or to two triangles; that is, whether we shall have one solution or two solutions from the given parts.*

From the given parts we may find the *two sides* and an *angle opposite one of these sides* in the polar triangle (Ch. Arts. 69 and 70, VIII.). There will be as many solutions for the triangle whose parts are given as for the polar triangle. But we have already considered the case of the triangle of which two sides and an angle opposite one of these sides are given (Art. 158). By passing to the polar triangle we can then find (under Case IV. and Art. 159) how many solutions it has, and thus determine how many solutions the triangle under this case has.

We can also ascertain this independently by referring to Ch. 26, VIII., and to Art. 137 of this book.

Thus, suppose the given parts are the angles A and B, and the side a.

The sum of the angles may be, 1. 180°; 2. $<180°$; or, 3. $>180°$.

Under these three heads, if $A=B$, $a=b$ (Ch. 24, VIII.), and there is *one solution*.

In the following cases we shall suppose A and B unequal.

1. If $A+B=180°$, $a+b$ also equals 180° (Art. 137), and there can be but *one solution.*

2. Let $A+B$ be $<180°$; and

(*a*) Let A be $>B$; there will be *one* solution; for if a is $<90°$, b must be $<a$ (Ch. 26, VIII.); and if a is $>90°$, b must be $<90°$, since $a+b<180°$ (Art. 137).

(*b*) Let A be $<B$, and $a<90°$; then there will be *two solutions*, for as A is $<B$, a is $<b$; but b may have two values, one $<90°$ but $>a$, and the other $>90°$, such that $a+b<180°$.

Thus, suppose $A+B<180°$, and $A<B$ and $a=40°$; b might be 60° or 120°, for with either value a is $<b$ and $a+b<180°$.

The case (under this head) in which A is $<B$ and $a>90°$, is *impossible*, for if A is $<B$, a is $<b$, $\therefore b$ is also $>90°$, that is, $a+b>180°$, which is impossible (Art. 137).

3. Let $A+B$ be $>180°$, and

(*a*) Let A be $<B$, then there will be *one* solution; for if $A<B$, $a<b$ (Ch. 26, VIII.); therefore if $a<90°$, b must be $>90°$, so that $a+b$ should be $>180°$ (Art. 137); or again, if $a>90°$, b must be $>90°$, because a is $<b$ as by hypothesis A is $<B$ (Ch. 26, VIII.).

(*b*) Let A be $>B$, and a be $>90°$; then there will be *two* solutions, for as A is $>B$, a is $>b$ (Ch. 26, VIII.); but b may have two values, one $<90°$, and therefore $<a$, or one $>90°$, but still $<a$, and $a+b>180°$ (Art. 137).

Thus, suppose $A+B>180°$, and a to be 120°, and $A>B$, then b might be 75° or 105°; with either value a is $>b$, and $a+b>180°$.

The case (under this head) in which A is $>B$, and a is $<90°$, is impossible; for as A is $>B$, a must also be $>b$; therefore b must be $<90°$, that is, $a+b$ is $<180°$, which is impossible (Art. 137).

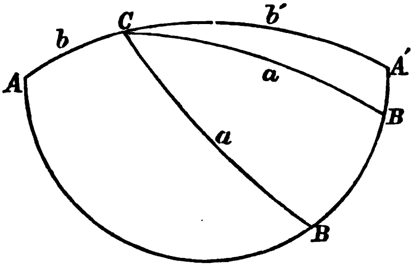
Therefore (considering only possible cases), when *two* *angles* of a spherical triangle and a *side opposite* one of them are given,

If the sum of the given angles is LESS *than* 180°, *and the angle opposite the given side is* LESS *than the other given angle* ((*b*) 2), or, *if the sum of the given angles is* GREATER *than* 180°, *and the angle opposite the given side is* GREATER *than the other given angle* ((*b*) 3), *there will be* TWO *solutions;* in *all* OTHER *cases only* ONE *solution.*

The two angles of a triangle are 125° and 110°, and the side opposite the first angle is 133°. It is required to solve the triangle.

Let $A = 125°$, $B = 110°$, $a = 133°$.

$A + B > 180°$, and A is $> B$; therefore there are *two* solutions.



$$\text{Sin. } b = \frac{\text{sin. } B}{\text{sin. } A} \text{ sin. } a = \frac{\text{sin. } 110°}{\text{sin. } 125°} \text{ sin. } 133°$$

$$\begin{aligned} \text{Log. sin. } 110° &= 9.972986 \\ \text{Log. sin. } 133° &= 9.864127 \\ \text{Ar. co. log. sin. } 125° &= 0.086635 \\ \hline \text{Log. sin. } 57°\ 1'\ 54.62'' &= 9.923748 \end{aligned}$$

$$b = CA = 57°\ 1'\ 54.62'' \qquad \tfrac{1}{2}(A+B) = 117°\ 30'$$

$$b' = CA' = 122°\ 58'\ 5.38'' \qquad \tfrac{1}{2}(A-B) = 7°\ 30'$$

To solve ABC

$$\tfrac{1}{2}(a+b) = 95°\ 0'\ 57.31''$$

$$\tfrac{1}{2}(a-b) = 37°\ 59'\ 2.69''$$

$$\text{Tan. } \tfrac{1}{2}\,c = \frac{\cos.\,\frac{1}{2}(A+B)}{\cos.\,\frac{1}{2}(A-B)}\tan.\,\tfrac{1}{2}(a+b) = \frac{\cos.\ 117^\circ\ 30'}{\cos.\ 7^\circ\ 30'}\tan.\ 95^\circ\ 0'\ 57.31''$$

$$\begin{aligned}
\text{Log. cos. } 117^\circ\ 30' &= 9.664406\\
\text{Log. tan. } 95^\circ\ 0'\ 57.31'' &= 11.056661\\
\text{Ar. co. log. cos. } 7^\circ\ 30' &= 0.003731\\
\hline
\text{Log. tan. } 79^\circ\ 19'\ 39.48'' &= 10.724798\\
c = AB &= 158^\circ\ 39'\ 18.96''.
\end{aligned}$$

$$\text{Cot. } \tfrac{1}{2}\,C = \frac{\cos.\,\frac{1}{2}(a+b)}{\cos.\,\frac{1}{2}(a-b)}\tan.\,\tfrac{1}{2}(A+B) = \frac{\cos.\ 95^\circ\ 0'\ 57.31''}{\cos.\ 37^\circ\ 59'\ 2.69''}\tan.\ 117^\circ\ 30'$$

$$\begin{aligned}
\text{Log. cos. } 95^\circ\ 0'\ 57.31'' &= 8.941673\\
\text{Log. tan. } 117^\circ\ 30' &= 10.283523\\
\text{Ar. co. log. cos. } 37^\circ\ 59'\ 2.69'' &= 0.103374\\
\hline
\text{Log. cot. } 77^\circ\ 58'\ 14.08'' &= 9.328570\\
ACB &= 155^\circ\ 56'\ 28.16''.
\end{aligned}$$

To solve $A'B'C$

$$\begin{aligned}
\tfrac{1}{2}(a+b') &= 127^\circ\ 59'\ 2.69''\\
\tfrac{1}{2}(a-b') &= 5^\circ\ 0'\ 57.31''
\end{aligned}$$

$$\text{Tan. } \tfrac{1}{2}\,c' = \frac{\cos.\ 117^\circ\ 30'}{\cos.\ 7^\circ\ 30'}\tan.\ 127^\circ\ 59'\ 2.69''$$

$$\begin{aligned}
\text{Log. cos. } 117^\circ\ 30' &= 9.664406\\
\text{Log. tan. } 127^\circ\ 59'\ 2.69'' &= 10.107439\\
\text{Ar. co. log. cos. } 7^\circ\ 30' &= 0.003731\\
\hline
\text{Log. tan. } 30^\circ\ 48'\ 50.62\tfrac{1}{2}'' &= 9.775576\\
c' = A'B' &= 61^\circ\ 37'\ 41.25''.
\end{aligned}$$

$$\text{Cot. } \tfrac{1}{2}\,C' = \frac{\cos.\ 127^\circ\ 59'\ 2.69''}{\cos.\ 5^\circ\ 0'\ 57.31''}\tan.\ 117^\circ\ 30'$$

$$\begin{aligned}
\text{Log. cos. } 127^\circ\ 59'\ 2.69'' &= 9.789187\\
\text{Log. tan. } 117^\circ\ 30' &= 10.283523\\
\text{Ar. co. log. cos. } 5^\circ\ 0'\ 57.31'' &= 0.001667\\
\hline
\text{Log. cot. } 40^\circ\ 7'\ 3.26'' &= 10.074377\\
A'C'B' &= 80^\circ\ 14'\ 6.52''.
\end{aligned}$$

(Check.) $$\frac{\sin.\ B}{\sin.\ b} = \frac{\sin.\ C}{\sin.\ c} = \frac{\sin.\ C'}{\sin.\ c'}$$

$$\begin{aligned}
\text{Log. sin. } 110^\circ &= 9.972986 &\quad \text{Log. sin. } 155^\circ\ 56'\ 28.16'' &= 9.610314\\
\text{Log. sin. } 57^\circ\ 1'\ 54.62'' &= 9.923748 &\quad \text{Log. sin. } 158^\circ\ 39'\ 18.96'' &= 9.561076\\
\hline
&\ \ \ 0.049238 & &\ \ \ 0.049238
\end{aligned}$$

$$\begin{array}{lr} \text{Log. sin. } 80^\circ\ 14'\ \ 6.52'' = & 9.993662 \\ \text{Log. sin. } 61^\circ\ 37'\ 41.25'' = & 9.944424 \\ \hline & 0.049238 \end{array}$$

Solve a spherical triangle when there are given:

EXAMPLE 1. Two angles, 55° 2′, 68° 10′; the side opposite the first angle, 48° 42′.

Ans. Sides, 58° 4′ 58.6″, 65° 26′ 30.6″; angle, 84° 5′ 12.4″;
or 121° 55′ 1.4″, 160° 12′ 23.8″; or 158° 15′ 51.1″.

2. Two angles, 150° 22′, 124° 12′; the side opposite the first angle 149° 20′.

Ans. Sides, 58° 33′ $28\frac{1}{3}''$, 143° 33′ 22″; angle, 144° 50′ 21.8″;
or 121° 26′ $31\frac{2}{3}''$, 73° 18′ 35.6″; or 111° 47′ 4.9″.

3. Two angles, 125°, 130°; the side opposite the first angle, 114°.

Ans. Sides, 121° 18′ 56.2″, 98° 36′ 17.2″; angle, 117° 33′ 11.7″.

4. Two angles, 12°, 25°; the side opposite the second angle, 60°.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES BY MEANS OF RIGHT-ANGLED TRIANGLES.

163. The preceding cases can all be solved by dividing the given triangle into two right-angled triangles, by drawing a great circle arc from one of the angles to the opposite side, or to the opposite side produced, and by then solving the two right-angled triangles.

Care should be taken, in drawing the perpendicular, to make one of the right-angled triangles formed contain two of the three given parts. (Art. 116.)

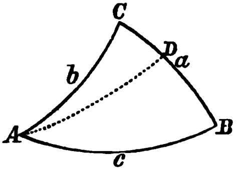
CASE I.—*The three sides being given.*

From the angle opposite the longest side draw a perpendicular to that side. The side will be thus divided internally into two segments. The half difference of the two segments can be found by formula (*a*) of Art. 142. This half difference added to the half side will give the greater segment; and subtracted from the half side will give the less segment. We shall then have, in each right-angled triangle, the hypotenuse and a side given to find the other parts. (Art. 120.)

Case II.—*The three angles being given.*

Find the sides of the polar triangle (Ch. Arts. 69 and 70, VIII.); solve that by the method of the preceding case; the supplements of the angles found will be the required sides of the given triangle.

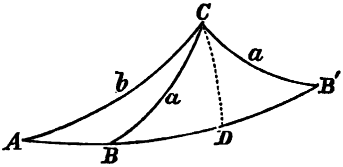
Case III.—*Two sides and an included angle being given.*



Suppose the given parts to be a, b, and C. Draw the perpendicular AD. Then in the triangle ACD we have the hypotenuse and an angle, C, to find CD (Art. 121). $BD = a - CD$. Then c can be found by 2, Art. 140. A and B can be found by Art. 138. The three last parts can also be found by finding all the parts of the two triangles ACD, BCD.

Case IV.—*Two sides and an angle opposite one of these sides being given.*

Suppose the given parts to be a, b, and A. Draw the perpendicular from C upon c or c produced.

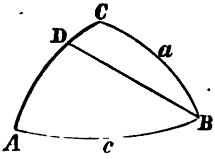


The figure represents one of the examples in which there are two solutions (see rule at the end of Art. 159). We shall only give one solution, as the method is the same in both solutions.

To solve the triangle ACD we have given the hypotenuse b and the angle A (Art. 121). BD can be found by 2, Art. 140. Then to solve the triangle BCD we have given the hypotenuse a and the side BD (Art. 120).

CASE V.—*Two angles and an included side being given.*

Suppose the given parts to be A, B, and c. Draw an arc from B perpendicular to b.

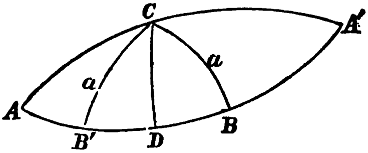


To solve the triangle ABD we have the hypotenuse c and the angle A (Art. 121). This will enable us to find BD and the angle ABD. If BD falls within the triangle, $CBD=B-ABD$; if without the triangle, $CBD=ABD-B$. Then to solve CBD we have given the side BD and the angle CBD (Art. 123).

CASE VI.—*Two angles and a side opposite one of these being given.*

(See rule at end of Art. 162.)

Suppose the given parts to be A, B, and a. Draw the perpendicular from C upon c (or upon c produced).



To solve the triangle CBD we have given the hypotenuse a and the adjacent angle (Art. 121). Having found BD, AD can be found by 1, Art. 140. AC can then be found by 2, 140. Or, by solving the triangle CBD we can find CD. Then to solve the triangle

ACD we have given a side, CD, and the opposite angle A (Art. 124). C will be the *sum* of ACD and BCD and c will be the *sum* of AD and BD, *if the perpendicular* falls *within* the triangle; but C will be the *difference* of ACD and $B'CD$, and c will be the *difference* of AD and DB', if the perpendicular falls *without* the triangle.

164. It is well to solve oblique-angled triangles by right-angled triangles when there is any difficulty in arriving at a true solution by means of formulas; for instance, when in Case I. s is very near 180°, or $s-a$, $s-b$, or $s-c$ is very near 0°; when in Case II. S is very near 90° or 270°, or $S-A$, $S-B$, or $S-C$ is very near 90°; when in Cases III. and IV. $\frac{1}{2}(a+b)$ is very near 90°, or $\frac{1}{2}(a-b)$ is very near 0°; when in Cases V. and VI. $\frac{1}{2}(A+B)$ is very near 90°, or $\frac{1}{2}(A-B)$ is very near 0°.

It is sometimes advantageous to combine the two methods so as to check the work.

THE END.

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